A probabilistic analogue of the Fourier extension conjecture and an implication for the Kakeya conjecture

Eric T. Sawyer

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In 1917, Besicovitch constructed a set of Lebesgue measure zero in $\mathbb{R}^n$ that 'paradoxically' contains a line segment in every direction answering the Kakeya needle problem.

The Kakeya *conjecture* claims that one cannot further compress such a union of line segments into a 'smaller' set.

More precisely, the conjecture is that a measurable set in $\mathbb{R}^n$ that contains a line segment in every direction, must have Hausdorff dimension n.

A classical argument shows that if the Fourier extension conjecture holds, i.e. functions $f \in L^p(\sigma_{n-1})$ have $\widehat{f\sigma_{n-1}} \in L^p(\mathbb{R}^n)$ for $p > \frac{2n}{n-1}$, then the Kakeya conjecture holds.

This argument uses Khintchine's inequality, which potentially weakens the extension conjecture that is needed, and leads to an $L^\infty(\sigma_{n-1})$ to $L^p(\lambda_n)$ estimate for a 'large' square function

$$\mathcal{S}_{\mathrm{large}}\widehat{f}(\xi) \equiv \left(\sum_{m=0}^{\infty} \sum_{\mathrm{caps}\ D_j : \ell(D_j) = 2^{-m}} \left| \widehat{f_{D_j}\sigma_{n-1}}(\xi) \right|^2 \right)^{\frac{1}{2}},$$

that is actually *equivalent* to the Kakeya maximal operator conjecture (which implies Kakeya).

Such square function estimates are equivalent to 'average' extension estimates by Khintchine's inequality.

One then searches for *smaller* square functions $\mathcal{S}_{\mathrm{small}}\widehat{f}$ whose boundedness remains equivalent to the operator variant of the Kakeya conjecture, in hopes that one can establish this boundedness directly.

This leads to the formulation of a *probabilistic* Fourier extension conjecture, which averages over certain 'wavelet' multipliers by ± 1, and when conjugations over unimodular functions are included, turns out to be equivalent to the Kakeya maximal operator conjecture.

In this talk we establish the simplest case of such a unimodular probabilistic extension, namely when the unimodular function is 1.

In order to prove boundedness of an operator $T : L^p(\sigma_{n-1}) \longrightarrow L^p(\lambda_n)$, there are at least two ways to begin, either by establishing a direct bound

$$\|Tf\sigma_{n-1}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\sigma_{n-1})},$$

i.e. a two weight norm inequality for T, or a duality bound

$$|\langle Tf\sigma_{n-1}, g\rangle| \lesssim \|f\|_{L^p(\sigma_{n-1})} \|g\|_{L^{p'}(\mathbb{R}^n)}.$$

Traditional methods of attacking the extension conjecture have used direct bounds and multilinear bounds.

Nazarov, Treil and Volberg used the duality bound approach in their pivotal theorem for the Hilbert transform T, and wrote f and g in weighted Haar expansions, which led to a large sum over pairs of dyadic cubes $\sum_{I,J} \langle T \triangle_I f, \triangle_J g \rangle$.

They then arranged these pairs according to scale and position into various subforms.

We will begin attacking the probabilistic extension conjecture in this way, but using smooth Alpert 'wavelet' decompositions with multiple vanishing moments in place of the Haar decompositions, resulting in some sort of analogue of the classical wave packet decompositions.

1. **Definitions and statements**

Overview of probabilistic Fourier extension and Kakeya

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Definitions and statements

Classical Alpert wavelets

- Denote by $L^2_{Q;\kappa}$ the finite dimensional subspace of L^2 defined by:

$$L^2_{Q;\kappa} \equiv \left\{ f = \sum_{Q' \in \mathfrak{C}(Q)} \mathbf{1}_{Q'} p_{Q';\kappa}(x) : \int_Q f(x)\, x_i^\ell dx = 0, \quad \text{for } 0 \leq \ell \leq \kappa -$$

where $p_{Q';\kappa}(x) = \sum_{\alpha \in \mathbb{Z}^n_+ : |\alpha| \leq \kappa - 1} a_{Q';\alpha} x^\alpha$ is a polynomial in $\mathbb{R}^n$ of degree $|\alpha| = \alpha_1 + ... + \alpha_n$ at most $\kappa - 1$.

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- Let $\left\{ h^a_{Q;\kappa} \right\}_{a \in \Gamma_{n,\kappa}}$ be an orthonormal basis for $L^2_{Q;\kappa}$ and let $\triangle_{Q;\kappa}$ denote orthogonal projection onto the finite dimensional subspace $L^2_{Q;k}$,

$$\triangle_{Q;\kappa} f \equiv \sum_{a \in \Gamma_{n,\kappa}} \left\langle f, h^a_{Q;\kappa} \right\rangle h^a_{Q;\kappa} , \qquad f \in L^2.$$

Smooth Alpert wavelets

convolution

- Given a small positive constant $\eta > 0$, define a smooth approximate identity by $\phi_{\eta}(x) \equiv \eta^{-n}\phi\left(\frac{x}{\eta}\right)$ where $\phi \in C_c^{\infty}(B_{\mathbb{R}^n}(0,1))$ has unit integral, $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$, and vanishing moments of *positive* order less than κ, i.e.

$$\int \phi(x)\, x^{\gamma} dx = \delta^0_{|\gamma|} = \begin{cases} 1 & \text{if} \quad |\gamma| = 0 \\ 0 & \text{if} \quad 0 < |\gamma| < \kappa \end{cases} . \tag{1}$$

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- In the spirit of symbol smoothing, we define *smooth* Alpert 'wavelets' by

$$h_{Q;\kappa}^{a,\eta} \equiv h_{Q;\kappa}^a * \phi_{\eta\ell(Q)}.$$

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- In the spirit of symbol smoothing, we define *smooth* Alpert 'wavelets' by

$$h^{a,\eta}_{Q;\kappa} \equiv h^a_{Q;\kappa} * \phi_{\eta\ell(Q)}.$$

- Then $h^a_{Q;\kappa}$ and $h^{a,\eta}_{Q;\kappa}$ coincide away from the η-neighbourhood (often referred to as a 'halo')

$$\mathcal{H}_\eta(Q) \equiv \{x \in \mathbb{R}^n : \mathrm{dist}(x, S_Q) < \eta\}, \tag{2}$$

of the skeleton $S_Q \equiv \bigcup_{Q' \in \mathfrak{C}_\mathcal{D}(Q)} \partial Q'$, where they are polynomials.

Theorem

Let $n, \kappa \in \mathbb{N}$ and $\eta > 0$ be sufficiently small depending on n and κ. Then there are a bounded invertible linear map $S_{\kappa,\eta} : L^p \to L^p$ $(1 < p < \infty)$ satisfying

$$\left\| \mathrm{Id} - S_{\kappa,\eta} \right\|_{L^p \to L^p} \leq C_{n,p}\eta \ , \tag{3}$$

and 'wavelets' $\left\{ h^a_{I;\kappa} \right\}_{I\in\mathcal{D},\ a\in\Gamma_n}$ and $\left\{ h^{a,\eta}_{I;\kappa} \right\}_{I\in\mathcal{D},\ a\in\Gamma_n}$ (with Γ_n a finite index set depending only on κ and n), and corresponding pseudoprojections $\left\{ \triangle_{I;\kappa} \right\}_{I\in\mathcal{D}}$ and $\left\{ \triangle^{\eta}_{I;\kappa} \right\}_{I\in\mathcal{D}}$ defined by

$$\triangle_{I;\kappa} f \equiv \sum_{a\in\Gamma_n} \left\langle f, h^a_{I;\kappa} \right\rangle h^a_{I;\kappa} \text{ and } \triangle^{\eta}_{I;\kappa} f \equiv \sum_{a\in\Gamma_n} \left\langle \left(S_{\kappa,\eta} \right)^{-1} f, h^a_{I;\kappa} \right\rangle h^{a,\eta}_{I;\kappa} \ ,$$

satisfying:

Smooth Alpert wavelets

Properties of wavelets I

- the standard properties,

$$\begin{aligned}
\left\| h^{a}_{I;\kappa} \right\|_{L^2} &= \left\| h^{a,\eta}_{I;\kappa} \right\|_{L^2} = 1, \\
\operatorname{Supp} h^{a}_{I;\kappa} &\subset I \text{ and } \operatorname{Supp} h^{a,\eta}_{I;\kappa} \subset (1+\eta)\, I, \\
\left\| \nabla^m h^{a,\eta}_{I;\kappa} \right\|_{\infty} &\leq C_m \left(\frac{1}{\eta \ell(I)} \right)^m \frac{1}{\sqrt{|I|}}, \qquad \text{for all } m \geq 0, \\
\int h^{a}_{I;\kappa}(x)\, x^{\alpha} dx &= \int h^{a,\eta}_{I;\kappa}(x)\, x^{\alpha} dx = 0, \qquad \text{for all } 0 \leq |\alpha| < \kappa.
\end{aligned} \tag{4}$$

- the standard properties,

$$\begin{aligned} \left\| h_{I;\kappa}^{a} \right\|_{L^2} &= \left\| h_{I;\kappa}^{a,\eta} \right\|_{L^2} = 1, \\ \operatorname{Supp} h_{I;\kappa}^{a} &\subset I \text{ and } \operatorname{Supp} h_{I;\kappa}^{a,\eta} \subset (1+\eta) I, \\ \left\| \nabla^m h_{I;\kappa}^{a,\eta} \right\|_{\infty} &\leq C_m \left(\frac{1}{\eta \ell (I)} \right)^m \frac{1}{\sqrt{|I|}}, \qquad \text{for all } m \geq 0, \\ \int h_{I;\kappa}^{a} (x) \, x^{\alpha} dx &= \int h_{I;\kappa}^{a,\eta} (x) \, x^{\alpha} dx = 0, \qquad \text{for all } 0 \leq |\alpha| < \kappa. \end{aligned} \tag{4}$$

- and for each $a \in \Gamma_n$ the wavelets $h_{I;\kappa}^{a}$ and $h_{I;\kappa}^{a,\eta}$ are translations and L^2-dilations of the unit wavelets $h_{Q_0;\kappa}^{a}$ and $h_{Q_0;\kappa}^{a,\eta}$ respectively, where $Q_0 = [0,1)^n$ is the unit cube in $\mathbb{R}^n$,

$$|I|^{\frac{1}{2}} h_{I;\kappa}^{a} = |Q_0|^{\frac{1}{2}} h_{Q_0;\kappa}^{a} \circ \varphi_I \text{ and } |I|^{\frac{1}{2}} h_{I;\kappa}^{a,\eta} = |Q_0|^{\frac{1}{2}} h_{Q_0;\kappa}^{a,\eta} \circ \varphi_I \, , \tag{5}$$

where $\varphi_I : I \to Q_0$ is the affine map taking I one-to-one and onto Q_0,

- and for all $1 < p < \infty$,

$$f = \sum_{I \in \mathcal{D},\ a \in \Gamma_n} \triangle_{I;\kappa}^{a} f = \sum_{I \in \mathcal{D},\ a \in \Gamma_n} \triangle_{I;\kappa}^{a,\eta} f, \qquad \text{for } f \in L^p \cap L^2,$$

$$\left\| \left(\sum_{I \in \mathcal{D},\ a \in \Gamma_n} \left| \triangle_{I;\kappa}^{a} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mu)} \approx \left\| \left(\sum_{I \in \mathcal{D},\ a \in \Gamma_n} \left| \triangle_{I;\kappa}^{a,\eta} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mu)} \approx$$

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- and for all $1 < p < \infty$,

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for $f \in L^p \cap L^2$,

- and for all $I \in \mathcal{D}$,

$$h^{a}_{Q;\kappa}(x) = h^{a,\eta}_{Q;\kappa}(x), \qquad \text{for } x \in \mathbb{R}^n \setminus \mathcal{H}_\eta(Q),$$

where $\mathcal{H}_\eta(Q)$ is the η-halo of the skeleton of Q.

Smooth Alpert wavelets

Properties of wavelets III

- and finally, the unsmoothed operators $\triangle_{I;\kappa}$ are self-adjoint orthogonal projections,

$$\triangle_{I;\kappa}\triangle_{J;\kappa} = \begin{cases} \triangle_{I;\kappa} & \text{if} \quad I = J \\ 0 & \text{if} \quad I \neq J \end{cases} . \tag{7}$$

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- The smoothed operators $\triangle_{I;\kappa}^{\eta}$ are neither self-adjoint, projections nor orthogonal, but come close.

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- The smoothed operators $\triangle^{\eta}_{I;\kappa}$ are neither self-adjoint, projections nor orthogonal, but come close.
- This theorem shows that the collection of 'almost' L^2 projections $\left\{\triangle^{\eta,a}_{I;\kappa}\right\}_{I\in\mathcal{D},\ a\in\Gamma_n}$ is a 'frame' for the Banach space L^p, $1 < p < \infty$.

Fourier extension conjecture

definition

- The Fourier extension conjecture is

$$\left(\int_{\mathbb{R}^n} \left|\mathcal{F}\left(f\sigma_{n-1}\right)(\xi)\right|^q d\xi\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{S}^{n-1}} |f(x)|^p \, d\sigma_{n-1}(x)\right)^{\frac{1}{p}}, \tag{8}$$

for all appropriate pairs $1 \leq p, q < \infty$, where σ_{n-1} is surface measure on the sphere $\mathbb{S}^{n-1}$, and $\mathcal{F}(\mu) \equiv \int_{\mathbb{R}^n} e^{-ix\cdot\xi} d\mu(x)$ denotes the Fourier transform of the measure μ.

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- The following special case is essentially equivalent:

$$\int_{\mathbb{R}^n} \left|\mathcal{F}\left(f\sigma_{n-1}\right)(\xi)\right|^p d\xi \leq C \int_{\mathbb{S}^{n-1}} \left|f(x)\right|^p d\sigma_{n-1}(x), \quad p > 2 + \frac{2}{n-1}.$$

Fourier extension conjecture

progress rectangle

$(0, \frac{1}{2})$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	**C**	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$(1, \frac{1}{2})$
$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$\times$	$\times$	$\times$	$\times$	$\times$	**A**	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
				$\cdot$			**B**	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
			$\cdot$						$\cdot$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
		$\cdot$									$\cdot$	$\times$	$\times$	$\times$	$\times$
	$\cdot$												$\cdot$	$\times$	$\times$
$(0, 0)$															$(1, 0)$

$$\mathbf{A} = \left(\frac{n-1}{2n}, \frac{n-1}{2n} \right) \text{ and } \mathbf{B} = \left(\frac{1}{2}, \frac{n-1}{2n+2} \right) \text{ and } \mathbf{C} = \left(\frac{1}{2}, \frac{1}{2} \right)$$

Contributions

- Let $\rho_n \equiv \frac{2n}{n-1} = 2 + \frac{2}{n-1}$.

$$q > \rho_n + \frac{2n}{n-1} \text{ (Stein 1967)},$$
$$q > \rho_2, \text{ for } n = 2 \text{ (Fefferman 1970; Carleson, Sjölin 1972; Zygmund}$$
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$$q > \rho_n + \frac{2}{n-1} \text{ (Tao 2003)},$$
$$q > 2 + \frac{12}{4n-3} \text{ if } n \equiv 0;\ 2 + \frac{3}{n-1} \text{ if } n \equiv 1;\ 2 + \frac{6}{2n-1} \text{ if } n \equiv 2 \text{ (}$$
$$\text{(Bourgain, Guth 2018)}.$$

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$$\text{(Bourgain, Guth 2018)}.$$

- Additional names: C. Demeter, J. Hickman, K. M. Rogers, C. Muscaru, I. Oliveira, J. Bennett, T. Carbery, A. Iosevich, R. Zhang, I. Laba, H. Wang who has the best result to date in $\mathbb{R}^3$, $p > 3 + \frac{3}{14}$.

Fourier extension conjecture

Parameterization of the sphere

- Let $\Phi(x) \equiv \left(x, \sqrt{1 - |x|^2}\right) \in \mathbb{S}^{n-1}$ be the standard parametization of the northern hemisphere of $\mathbb{S}^{n-1}$. Let S be a cube of side length 1 centered at the origin in $\mathbb{R}^{n-1}$ and define

$$T_S f(\xi) \equiv \int_S e^{-i\Phi(x)\cdot\xi} f(x)\, dx, \qquad \xi \in \mathbb{R}^n,$$

so that $T_S f = \mathcal{F}\Phi_* (f\lambda_{n-1})$ where $\Phi_*\nu$ denotes the pushforward of a measure ν under the map Φ.

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- Then the Fourier extension problem is equivalent to boundedness of T_S,

$$\|T_S f\|_{L^q(\lambda_n)} \leq C\, \|f\|_{L^p(S)}\, .$$

- Let $\{\triangle_{I;\kappa}\}_{I\in\mathcal{D}:\ I\subset S}$ be the family of Alpert projections $\triangle_{I;\kappa}=\sum_{a\in\Gamma_n}\left\langle f,h_{I;\kappa}^{a}\right\rangle h_{I;\kappa}^{a}$ on $L^2(S)$ as in Theorem 1.

Fourier extension conjecture

involutive smooth Alpert mulitpliers

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- For $\mathbf{a}=\left\{a_I\right\}_{I\in\mathcal{D}}\in\left\{1,-1\right\}^{\mathcal{D}}$ and $f\in L^p\left(S\right)$, define the *involutive Alpert multiplier* $\mathcal{A}_{\mathbf{a}}$ by

$$\mathcal{A}_{\mathbf{a}}f\equiv\sum_{I\in\mathcal{D}}a_I\,\triangle_{I;\kappa}^{\eta}f,$$

which is $\sum_{I\in\mathcal{D}}\pm\triangle_{I;\kappa}^{\eta}f$ for a choice of $\pm$ determined by $\mathbf{a}$.

Fourier extension conjecture

involutive smooth Alpert mulitpliers

- Let$\left\{\triangle_{I;\kappa}\right\}_{I\in\mathcal{D}:\ I\subset S}$ be the family of Alpert projections $\triangle_{I;\kappa}=\sum_{a\in\Gamma_n}\left\langle f,h^a_{I;\kappa}\right\rangle h^a_{I;\kappa}$ on $L^2\left(S\right)$ as in Theorem 1.
- For $\mathbf{a}=\left\{a_I\right\}_{I\in\mathcal{D}}\in\left\{1,-1\right\}^{\mathcal{D}}$ and $f\in L^p\left(S\right)$, define the *involutive Alpert multiplier* $\mathcal{A}_{\mathbf{a}}$ by

$$\mathcal{A}_{\mathbf{a}}f\equiv\sum_{I\in\mathcal{D}}a_I\,\triangle^{\eta}_{I;\kappa}f,$$

  which is $\sum_{I\in\mathcal{D}}\pm\triangle^{\eta}_{I;\kappa}f$ for a choice of $\pm$ determined by $\mathbf{a}$.
- Let $S_{\kappa,\eta}$ be the bounded invertible linear map on L^p given in Theorem 1, that takes Alpert wavelets $h^a_{I;\kappa}$ to their smooth counterparts $h^{a,\eta}_{I;\kappa}=h^a_{I;\kappa}*\phi_{\eta\ell(I)}$.

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- For $\mathbf{a} = \{a_I\}_{I\in\mathcal{G}} \in \{1,-1\}^{\mathcal{G}}$ and $f \in L^p(S)$, define the *involutive* **smooth** *Alpert multiplier* $\mathcal{A}^{S_{\kappa,\eta}}_{\mathbf{a}}$ by conjugation with the bounded invertible map $S_{\kappa,\eta}$, i.e.

$$\mathcal{A}^{S_{\kappa,\eta}}_{\mathbf{a}} f \equiv S_{\kappa,\eta} \mathcal{A}_{\mathbf{a}} S^{-1}_{\kappa,\eta} f.$$

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probabilistic version

- Then we identify $2^{\mathcal{G}}$ and $\{1,-1\}^{\mathcal{G}}$ and equip $2^{\mathcal{G}}$ with the probability measure μ that satisfies,

$$\mu_{\Gamma}(E) \equiv \mu\left(\left\{E \mid E \subset 2^{\Gamma}\right\}\right) = \frac{|E|}{|2^{\Gamma}|}, \qquad E \subset 2^{\Gamma} \text{ with } \Gamma \subset \mathcal{G} \text{ finite.}$$

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- We define the expectation operator $\mathbb{E}^{\mu}_{2^{\mathcal{G}}}$ by $\mathbb{E}^{\mu}_{2^{\mathcal{G}}} F \equiv \int_{2^{\mathcal{G}}} F(\mathbf{a})\, d\mu(\mathbf{a})$ for F a nonnegative function on $2^{\mathcal{G}}$.

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- The *probabilistic* extension problem is the inequality,

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which asks, roughly speaking, if the extension inequality (8) holds when averaged over all involutive smooth Alpert multipliers.

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  which asks, roughly speaking, if the extension inequality (8) holds when averaged over all involutive smooth Alpert multipliers.
- The probabilistic analogue (9) fails for the same pairs (p,q) that (8) is currently known to fail for.

Probabilistic Fourier extension theorem

- Our probabilistic analogue of the Fourier extension conjecture is,

$$\mathbb{E}^{\mu}_{2^{\mathcal{G}}} \left\| T_S \mathcal{A}^{S_{\kappa,\eta}}_{\mathbf{a}} f \right\|_{L^p(\lambda_n)} \lesssim \|f\|_{L^p(S)} , \qquad \text{if and only if } \frac{2n}{n-1} < p \leq \infty. \tag{10}$$

Theorem (Probabilistic extension conjecture)

The probabilistic Fourier extension inequality (10) holds in all dimensions $n \geq 2$.

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Theorem (Probabilistic extension conjecture)

The probabilistic Fourier extension inequality (10) holds in all dimensions $n \geq 2$.

- **Conjecture:** For all unimodular Ψ on S, there holds the *unimodular probabilistic Fourier extension* inequality uniformly in conjugations $\mathcal{A}^{\Psi S_{\kappa,\eta}}_{\mathbf{a}} = \Psi S_{\kappa,\eta} \mathcal{A}_{\mathbf{a}} \left(\Psi S_{\kappa,\eta} \right)^{-1}$ of Alpert multipliers $\mathcal{A}_{\mathbf{a}}$,

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- If $S \subset \mathbb{R}^n$ and $d \in [0, \infty)$, define

$$H_\delta^d(S) \equiv \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(B_i)^d : S \subset \bigcup_{i=1}^{\infty} B_i \text{ and } \operatorname{diam}(B_i) < \delta \right\},$$

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$$\limsup_{\delta \searrow 0} \frac{\ln N(S,\delta)}{\ln \frac{1}{\delta}} \text{ and } \liminf_{\delta \searrow 0} \frac{\ln N(S,\delta)}{\ln \frac{1}{\delta}},$$

respectively where $N(S, \delta)$ is the minimal number of sets of diameter at most δ needed to cover S.

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- For $\delta > 0$, $\omega \in \mathbb{S}^{n-1}$ and $a \in \mathbb{R}^n$, let $T_\delta^\omega(a)$ denote the tube in $\mathbb{R}^n$, centered at a and oriented in the direction of ω, and which has length 1 in that direction and cross-sectional radius $\delta > 0$. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, define the *Kakeya maximal operator* by

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Equivalence of conjectures

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$$\mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\| T_S \mathcal{A}_{\mathbf{a}}^{\Psi S_{\kappa,\eta}} f \right\|_{L^p(\lambda_n)} \lesssim \left\| f \right\|_{L^p(S)}, \qquad \text{if } \frac{2n}{n-1} < p \leq \infty,$$

where $S_{\kappa,\eta} \mathcal{A}_{\mathbf{a}} \left(S_{\kappa,\eta} \right)^{-1}$ is a random smooth Alpert multiplier, and $\mathcal{A}_{\mathbf{a}}^{\Psi S_{\kappa,\eta}} = \Psi S_{\kappa,\eta} \mathcal{A}_{\mathbf{a}} \left(\Psi S_{\kappa,\eta} \right)^{-1}$ is a conjugation by a unimodular function Ψ on S.

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- The implication "unimodular probabilistic" implies "Kakeya maximal" follows upon appealing to the large square function $\mathcal{S}_{\text{large}} T_S f$, whose $L^\infty \to L^p$ boundedness, $p > \frac{2n}{n-1}$, is equivalent to "Kakeya maximal" by classical arguments dating back to Bourgain.

Kakeya maximal operator implication

Theorem (Kakeya maximal operator implication)

Let $n < p \leq \infty$. If the probabilistic Fourier extension inequality holds uniformly over conjugations $\mathcal{A}_{\mathbf{a}}^{\Psi S_{\kappa,\eta}}$ of Alpert multipliers for all unimodular functions Ψ, then the Kakeya maximal operator $f_\delta^*(\omega)$ satisfies

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Theorem (Kakeya conjecture)

Suppose the Kakeya maximal operator bound above holds. Then the Kakeya conjecture holds: any measurable subset of $\mathbb{R}^n$, $n \geq 1$, that contains a unit line segment in every direction, has Hausdorff dimension n.

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- I am indebted to Hong Wang and Ruixiang Zhang for pointing out serious gaps in earlier versions of this paper.

Discussion of proofs

Proof of the Kakeya maximal implication

- We show that the unimodular probabilistic extension conjecture implies the *dual* form of the Kakeya maximal operator conjecture, which is this:

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$$\left\| \sum_{T \in \mathbb{T}} \mathbf{1}_T \right\|_{L^p} \leq C_{n,p,\varepsilon} \delta^{\frac{n-1}{p} - \varepsilon} \left(\# \mathbb{T} \right)^{\frac{1}{p}}, \qquad \text{for all } \delta, \varepsilon > 0,$$

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  which is essentially a restricted weak type inequality for a discretization.
- A nondegeneracy inequality is needed to show that the Fourier transform of a smooth Alpert wavelet $\Phi_* h_{I;\kappa}^{\eta}$ on the surface of the sphere behaves, on the associated uncertainty tube T, like that of a δ-cap on the sphere, where $\delta = \ell(I)$.

- We write

$$\langle T_S f, g \rangle = \sum_{(I,J) \in \mathcal{G} \times \mathcal{D}} \left\langle T_S \, \triangle_{I;\kappa}^{n-1,\eta} f, \triangle_{J;\kappa}^{n,\eta} g \right\rangle ,$$

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- If there is sufficient *oscillation* in either I or J in the inner product, we can integrate by parts against the smooth Alpert wavelet, or use stationary phase, while if there is sufficient *smoothness* in either I or J, we can exploit the high order moment vanishing of the smooth Alpert wavelets. In all such cases there is sufficient geometric decay in the off-diagonal inner products to add up.

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- What remains are the *resonant* inner products where neither oscillation nor smoothness is sufficiently present.

When the inner products $\left\langle \triangle_I \widehat{f\sigma_{n-1}}, \triangle_J g \right\rangle$ are resonant, both sides of the duality experience exponentials whose wavelengths are roughly the side length of the cube, and neither integration by parts, moment vanishing nor stationary phase are of much use.
To circumvent this problem, we apply Hölder's inequality and turn instead to the direct method of establishing $\left\| \widehat{f\sigma_{n-1}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\sigma_{n-1})}$ for $p > \frac{2n}{n-1}$, but where f is now a relatively simple smooth Alpert 'projection' at a single scale. Then we interpolate between L^2 and L^4 estimates for these simple 'projections'.

Unfortunately, at each scale the L^2 bound is large and the L^4 bound is not small, yielding no geometric decay in scales to sum.
However, if one averages over involutive smooth multipliers, then the L^4 bound collapses, due to cancellation in a substantial proportion of the off diagonal terms, and results in a geometric decay that is exactly what is needed to obtain boundedness of the averaged extension operator when $p > \frac{2n}{n-1}$.

Lemma

Let $n \geq 2$. Assume that

$$\left\| \widehat{f_{\Phi,2m}} \right\|_{L^2\left(\left|\widehat{\varphi_m}\right|^2 \lambda_n\right)} \lesssim 2^{\frac{m}{2}} \left\| f \right\|_{L^2(\sigma_{n-1})}, \tag{11}$$

$$\left(\mathbb{E}^{\mu}_{2^{\mathcal{G}}} \left\| \widehat{f_{\Phi,2m}} \right\|^4_{L^4\left(\left|\widehat{\varphi_m}\right|^2 \lambda_n\right)} \right)^{\frac{1}{4}} \lesssim 2^{-m\frac{n-2}{4}} \left\| f \right\|_{L^4(\sigma_{n-1})}.$$

If $p > \frac{2n}{n-1}$, then there is $\varepsilon_{p,n} > 0$ such that

$$\left(\mathbb{E}^{\mu}_{2^{\mathcal{G}}} \left\| T_S \mathsf{Q}^{\eta}_m f \right\|^p_{L^p\left(\left|\widehat{\varphi_m}\right|^4 \lambda_n\right)} \right)^{\frac{1}{p}} \lesssim 2^{-m\varepsilon_{p,n}} \left\| f \right\|_{L^p(\sigma_{n-1})},$$

holds for every $m \in \mathbb{N}$ with implied constant independent of m. Without the expectation, the L^4 bound is just a constant with no decay.

Proof of the smooth Alpert wavelet theorem

- For the purposes of this proof we change notation by defining

$$\triangle_{I;\kappa}^{\eta} f \equiv \sum_{a \in \Gamma_n} \left\langle f, h_{I;\kappa}^{a} \right\rangle h_{I;\kappa}^{a,\eta} = \left(\triangle_{I;\kappa} f \right) * \phi_{\eta \ell(I)} \ .$$

Next we show that the linear map $S_{\kappa,\eta}$ defined by

$$S_{\kappa,\eta} f \equiv \sum_{I \in \mathcal{D},\ a \in \Gamma_n} \left\langle f, h_{I;\kappa}^{a} \right\rangle h_{I;\kappa}^{\eta,a} = \sum_{I \in \mathcal{D}} \triangle_{I;\kappa}^{\eta} f \ , \qquad f \in L^p, \tag{12}$$

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- and that we have the reproducing formula,

$$f(x) = S_{\kappa,\eta} \left(S_{\kappa,\eta}^{\mathcal{D}}\right)^{-1} f(x) = \sum_{I\in\mathcal{D},\ a\in\Gamma_n} \left\langle \left(S_{\kappa,\eta}^{\mathcal{D}}\right)^{-1} f, h_{I;\kappa}^{a} \right\rangle h_{I;\kappa}^{a,\eta}(x) , \quad f$$

with convergence in the L^p norm and almost everywhere.

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3. Replacing the sphere with any smooth surface of nonvanishing Gaussian curvature, and possibly with appropriate smooth surfaces of finite type (and with altered indices p),
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5. and finally to boundedness of the maximal spherical partial sum operator in a probabilistic sense.

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The Fourier extension and Kakeya conjectures have been open for decades. Some probabilistic questions:

1. Does the unimodular probabilistic Fourier extension conjecture hold? If so then the Kakeya conjecture would also hold.
2. A probabilistic analogue of the Bochner-Riesz conjecture or even the stronger local smoothing conjecture. In the context of the (nonprobabilistic) extension conjecture, see Sogge for a proof that local smoothing implies Bochner-Riesz, and Tao for a proof that Bochner-Riesz implies Fourier restriction,
3. Replacing the sphere with any smooth surface of nonvanishing Gaussian curvature, and possibly with appropriate smooth surfaces of finite type (and with altered indices p),
4. Multilinear probabilistic variants of the extension conjecture,
5. and finally to boundedness of the maximal spherical partial sum operator in a probabilistic sense.
6. **Thanks to the organizers Chun-Yen, Daniel and Cody!**