A PROBABILISTIC ANALOGUE OF THE FOURIER EXTENSION CONJECTURE

ERIC T. SAWYER†

Abstract. The Fourier extension conjecture in n dimensions is

$$\left\|T\mathbf{1}_{U_0}f\right\|_{L^p(\lambda_n)} \leq C\left\|f\right\|_{L^p\left(B_{n-1}\left(0,\frac{1}{2}\right)\right)} , \qquad p>\frac{2n}{n-1},$$

where $Tf(\xi)\equiv\int_{B_{n-1}\left(0,\frac{1}{2}\right)}e^{-i\Phi(x)\cdot\xi}f(x)\,dx$, $U_0\subset B_{n-1}\left(0,\frac{1}{2}\right)\subset\mathbb{R}^{n-1}$, $\Phi(x)=\left(x,\sqrt{1-|x|^2}\right)$ and λ_n is Lebesgue measure on $\mathbb{R}^n$. Noting that $f=\sum_{I\in\mathcal{G}}\triangle_{I;\kappa}^{\eta}f$, we prove that the following probabilistic analogue of the Fourier extension conjecture,

$$\left(\mathbb{E}_{2^{\mathcal{G}}}\left\|T\mathbf{1}_{U_0}\sum_{I\in\mathcal{G}}\pm\triangle_{I;\kappa}^{\eta}f\right\|_{L^p(\lambda_n)}^p\right)^{\frac{1}{p}}\leq C\left\|f\right\|_{L^p\left(B_{n-1}\left(0,\frac{1}{2}\right)\right)} ,$$

holds for all $f\in L^p\left(B_{n-1}\left(0,\frac{1}{2}\right)\right)$ if and only if $p>\frac{2n}{n-1}$. The operator $\mathbb{E}_{2^{\mathcal{G}}}$ averages over all sequences of ±1, where $\mathcal{G}$ is a grid of dyadic subcubes containing U, and where $\triangle_{I;\kappa}^{\eta}$ is a smooth Alpert pseudoprojection, resulting in a 'martingale tranform' analogue.

By Khintchine's inequalities, the probabilistic analogue of the Fourier extension conjecture is equivalent to the square function estimate,

$$\left\|\mathcal{S}_{T\mathbf{1}_{U_0}}f\right\|_{L^q(\lambda_n)}\lesssim\left\|f\right\|_{L^p\left(B_{n-1}\left(0,\frac{1}{2}\right)\right)}, \qquad \text{if and only if } \frac{2n}{n-1}<p\leq\infty,$$

where

$$\mathcal{S}_{T\mathbf{1}_{U_0}}f\equiv\left(\sum_{I\in\mathcal{G}}\left|T\mathbf{1}_{U_0}\triangle_{I;\kappa}^{n-1,\eta}f\right|^2\right)^{\frac{1}{2}}.$$

To prove this probabilistic analogue of the extension conjecture, we use frames for L^p consisting of smooth compactly supported Alpert wavelets having a large number $\kappa>\frac{n}{2}$ of vanishing moments, along with stationary phase and probabilistic interpolation of L^2 and L^4 estimates, as part of a two weight testing strategy pioneered by Nazarov, Treil and Volberg. We use probability to obtain L^4 estimates with the correct decay when dealing with resonant subforms, thus circumventing the most challenging issues arising in the Fourier extension conjecture.

Contents

Date: March 10, 2025.
† Research supported in part by a grant from the National Science and Engineering Research Council of Canada.

1. Introduction

In this paper we consider a probabilistic analogue of the Fourier extension problem

$$\left(\int_{\mathbb{R}^n}\left|\mathcal{F}\left(f\sigma_{n-1}\right)(\xi)\right|^q d\xi\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{S}^{n-1}}|f(x)|^p d\sigma_{n-1}(x)\right)^{\frac{1}{p}}, \tag{1.1}$$

for $1 \leq p,q < \infty$ and where σ_{n-1} is surface measure on the sphere $\mathbb{S}^{n-1}$, and $\mathcal{F}(\mu) \equiv \int_{\mathbb{R}^n} e^{-ix\cdot\xi} d\mu(x)$ denotes the Fourier transform of the measure μ.

1.1. **The probabilistic extension problem.** Let $\Phi(x) \equiv \left(x, \sqrt{1-|x|^2}\right) \in \mathbb{S}^{n-1}$ be the standard parametization of the northern hemisphere of $\mathbb{S}^{n-1}$. Let $B_{n-1}\left(0,\frac{1}{2}\right)$ be the ball of radius $\frac{1}{2}$ centered at the origin in $\mathbb{R}^{n-1}$, and define

$$Tf(\xi) \equiv \int_{B_{n-1}\left(0,\frac{1}{2}\right)} e^{-i\Phi(x)\cdot\xi} f(x) \frac{dx}{|\det \nabla\Phi(x)|}, \qquad \xi \in \mathbb{R}^n,$$

for $f \in L^p\left(B_{n-1}\left(0,\frac{1}{2}\right)\right)$. Thus $Tf = \mathcal{F}\Phi_*\left(f\lambda_{n-1}\right) = \Phi_*\widehat{\left(f\lambda_{n-1}\right)}$, where $\Phi_*\nu$ denotes the pushforward of a measure ν under the map Φ. Then the Fourier extension problem (1.1) is equivalent to boundedness of the operator $T\mathbf{1}_{U_0}$, i.e.

$$\left\|T\mathbf{1}_{U_0} f\right\|_{L^q(\lambda_n)} \leq C\|f\|_{L^p\left(B\left(0,\frac{1}{2}\right)\right)}, \tag{1.2}$$

for a fixed subcube U_0 of $B_{n-1}\left(0,\frac{1}{2}\right)$ (after considering finitely many rotations). The Jacobian $\frac{1}{|\det \nabla\Phi(x)|}$ is roughly 1 on $B\left(0,\frac{1}{2}\right)$ and can be absorbed into the function $f(x)$ - we will often abuse notation by simply ignoring it.

Now let $\left\{\triangle_{I;\kappa}^{n-1,\eta}\right\}_{I\in\mathcal{G}}$ be the family of smooth Alpert pseudoprojections

$$\triangle_{I;\kappa}^{n-1,\eta} = \sum_{a\in\Gamma_{n-1}} \left\langle \left(S_{\kappa,\eta}\right)^{-1} f, h_{I;\kappa}^{a}\right\rangle h_{I;\kappa}^{a,\eta}$$

on $L^2\left(\mathbb{R}^{n-1}\right)$ as given in Theorem 6 below, where $\mathcal{G}$ is a dyadic grid containing U_0. Then we can rewrite (1.2) as,

$$\left\|T\mathbf{1}_{U_0}\sum_{I\in\mathcal{G}} \triangle_{I;\kappa}^{n-1,\eta} f\right\|_{L^q(\lambda_n)} \leq C\|f\|_{L^p\left(B\left(0,\frac{1}{2}\right)\right)}. \tag{1.3}$$

The *probabilistic* Fourier extension problem is then to decide when the following 'martingale transform' analogue of (1.3) holds,

$$\mathbb{E}_{2^{\mathcal{G}}}^{\mu}\left\|T\mathbf{1}_{U_0}\sum_{I\in\mathcal{G}} \pm \triangle_{I;\kappa}^{n-1,\eta} f\right\|_{L^q(\lambda_n)} \leq C\|f\|_{L^p\left(B\left(0,\frac{1}{2}\right)\right)}, \tag{1.4}$$

where the expectation $\mathbb{E}_{2^{\mathcal{G}}}^{\mu}$ is taken over all choices of $\pm$ for each $I \in \mathcal{G}$. We point out that it is not hard to see that the probabilistic analogue (1.4) fails for the same pairs (p,q) that (1.1) is currently known to fail for.

By Khinchine's inequalities, (1.4) is equivalent to the square function estimate

$$\left\|\mathcal{S}_{T\mathbf{1}_{U_0}} f\right\|_{L^q(\lambda_n)} \lesssim \|f\|_{L^p\left(B\left(0,\frac{1}{2}\right)\right)}, \tag{1.5}$$

where $\mathcal{S}_{T\mathbf{1}_{U_0}}$ is the square function defined by

$$\mathcal{S}_{T\mathbf{1}_{U_0}} f \equiv \left(\sum_{I\in\mathcal{G}[U]} \left|T\mathbf{1}_{U_0} \triangle_{I;\kappa}^{n-1,\eta} f\right|^2\right)^{\frac{1}{2}}. \tag{1.6}$$

1.1.1. *A precise description of the martingale transform.* We begin with a more precise description of the 'martingale transform' inequality (1.4), and then establish a reduction to certain Alpert projections. Let $\mathcal{G}$ be a grid in $\mathbb{R}^{n-1}$, and let $\left\{\triangle_{I;\kappa}^{n-1}\right\}_{I\in\mathcal{G}}$ be the orthogonal family of Alpert projections $\triangle_{I;\kappa}^{n-1}=\sum_{a\in\Gamma_{n-1}}\left\langle f,h_{I;\kappa}^{n-1,a}\right\rangle h_{I;\kappa}^{n-1,a}$ on $L^2\left(\mathbb{R}^{n-1}\right)$ as in Theorem 6, and let $\left\{\triangle_{I;\kappa}^{n-1,\eta}\right\}_{I\in\mathcal{G}}$ be the frame of *smooth* Alpert pseudoprojections on $L^p\left(\mathbb{R}^{n-1}\right)$. For $\mathbf{a}=\{a_I\}_{I\in\mathcal{G}}\in\{1,-1\}^{\mathcal{G}}$ and $f\in L^p\left(\mathbb{R}^{n-1}\right)$, define the *Alpert martingale transform* $\mathcal{A}_{\mathbf{a}}$ by

$$\mathcal{A}_{\mathbf{a}}f\equiv\sum_{I\in\mathcal{G}}a_I\,\triangle_{I;\kappa}^{n-1}\,f,$$

which is $\sum_{I\in\mathcal{G}}\pm\triangle_{I;\kappa}\,f$ for a choice of $\pm$ determined by $\mathbf{a}$.

Given linear operators L and S with S invertible, define the conjugation of L by S as

$$L^S\equiv SLS^{-1}.$$

Let $S_{\kappa,\eta}$ be the bounded invertible linear map on L^p given in Theorem 6, that takes Alpert wavelets $h_{I;\kappa}^{n-1,a}$ to their smooth counterparts $h_{I;\kappa}^{n-1,a,\eta}=h_{I;\kappa}^{n-1,a}*\phi_{\eta\ell(I)}$. For $\mathbf{a}=\{a_I\}_{I\in\mathcal{G}}\in\{1,-1\}^{\mathcal{G}}$ and $f\in L^p\left(\mathbb{R}^{n-1}\right)$, define the ***smooth*** *Alpert martingale transform*

$$\mathcal{A}_{\mathbf{a}}^{S_{\kappa,\eta}}f\equiv\sum_{I\in\mathcal{G}}a_I\,\triangle_{I;\kappa}^{n-1,\eta}\,f=\sum_{I\in\mathcal{G}}\pm\triangle_{I;\kappa}^{n-1,\eta}\,f$$

by conjugating $\mathcal{A}_{\mathbf{a}}$ with the bounded invertible map $S_{\kappa,\eta}$, i.e.

$$\mathcal{A}_{\mathbf{a}}^{S_{\kappa,\eta}}f\equiv S_{\kappa,\eta}\mathcal{A}_{\mathbf{a}}S_{\kappa,\eta}^{-1}f=S_{\kappa,\eta}\sum_{I\in\mathcal{G}}a_I\left\langle S_{\kappa,\eta}^{-1}f,h_{I;\kappa}^{n-1}\right\rangle h_{I;\kappa}^{n-1}=\sum_{I\in\mathcal{G}}a_I\left\langle S_{\kappa,\eta}^{-1}f,h_{I;\kappa}^{n-1}\right\rangle h_{I;\kappa}^{n-1,\eta}=\sum_{I\in\mathcal{G}}a_I\,\triangle_{I;\kappa}^{n-1,\eta}\,f.$$

Note that both $\mathcal{A}_{\mathbf{a}}$ and $\mathcal{A}_{\mathbf{a}}^{S_{\kappa,\eta}}$ are involutions, $\mathcal{A}_{\mathbf{a}}^2=\left(\mathcal{A}_{\mathbf{a}}^{S_{\kappa,\eta}}\right)^2=\mathrm{Id}$.

Since we will be using the notation $L^{S_{\kappa,\eta}}$ for various operators $L=\mathcal{A}_{\mathbf{a}},\mathcal{A}_{\mathbf{a}}\mathsf{P}_S,\mathcal{A}_{\mathbf{a}}\mathsf{Q}_K^s$ etc., we declutter the exponent by writing

$$L^{\spadesuit}\equiv L^{S_{\kappa,\eta}},$$

when the bounded invertible linear operator is $S_{\kappa,\eta}$.

Then we identify $2^{\mathcal{G}}$ and $\{1,-1\}^{\mathcal{G}}$ and equip $2^{\mathcal{G}}$ with the probability measure μ that satisfies,

$$\mu_\Lambda(E)\equiv\mu\left(\left\{E\mid E\subset 2^\Lambda\right\}\right)=\frac{|E|}{|2^\Lambda|},\qquad E\subset 2^\Lambda\text{ with }\Lambda\subset\mathcal{G}\text{ finite},$$

where $|F|$ denotes cardinality of a finite subset of $\mathcal{G}$, and $\mu\left(\left\{E\mid E\subset 2^\Lambda\right\}\right)$ is the conditional probability of E given that $E\subset 2^\Lambda$ (here 2^Λ is a set of μ-measure zero, and see e.g. [Hyt] for a construction of such a measure μ). We define the expectation operator $\mathbb{E}_{2^{\mathcal{G}}}^{\mu}$ by

$$\mathbb{E}_{2^{\mathcal{G}}}^{\mu}F\equiv\int_{2^{\mathcal{G}}}F(\mathbf{a})\,d\mu(\mathbf{a})$$

for F a nonnegative function on $2^{\mathcal{G}}=\{1,-1\}^{\mathcal{G}}$, so that (1.4) becomes,

$$(1.7)\qquad\mathbb{E}_{2^{\mathcal{G}}}^{\mu}\left\|T\mathbf{1}_{U_0}(\mathcal{A}_{\mathbf{a}})^{\spadesuit}f\right\|_{L^q(\lambda_n)}=\mathbb{E}_{2^{\mathcal{G}}}^{\mu}\int_{2^{\mathcal{G}}}\left\|T\mathbf{1}_{U_0}(\mathcal{A}_{\mathbf{a}})^{\spadesuit}f\right\|_{L^q(\lambda_n)}d\mu(\mathbf{a})\leq C\|f\|_{L^p\left(B\left(0,\frac{1}{2}\right)\right)}.$$

1.1.2. *A reduction of the martingale transform inequality.* We now replace $\mathbf{1}_{U_0}(\mathcal{A}_{\mathbf{a}})^{\spadesuit}f=\mathbf{1}_{U_0}S_{\kappa,\eta}\mathcal{A}_{\mathbf{a}}S_{\kappa,\eta}^{-1}$ in (1.7) with

$$(\mathcal{A}_{\mathbf{a}}\mathsf{P}_U)^{\spadesuit}f\equiv S_{\kappa,\eta}\mathcal{A}_{\mathbf{a}}\mathsf{P}_U S_{\kappa,\eta}^{-1}=S_{\kappa,\eta}\mathcal{A}_{\mathbf{a}}\sum_{I\in\mathcal{G}[U]}\triangle_{I;\kappa}S_{\kappa,\eta}^{-1}f=\sum_{I\in\mathcal{G}[U]}a_I\,\triangle_{I;\kappa}^{\eta}\,f,$$

where $\mathsf{P}_U g\equiv\sum_{I\in\mathcal{G}[U]}\triangle_{I;\kappa}g$ is the Alpert projection of a function g in which the sum over cubes I is restricted to those contained in U. We claim that this new inequality is sufficient for (1.7) in the case

$$U=\pi_{\mathcal{G}}^{(2)}U_0$$

is the $\mathcal{G}$-grandparent of U_0, where we assume $3U_0\subset U$, i.e. U_0 is an *interior* grandchild of U.

More precisely, we will show in a moment that (1.7) is implied by the following *truncated* inequality,

$$\mathbb{E}^{\mu}_{2^{\mathcal{G}}}\left\|T\left(\mathcal{A}_{\mathbf{a}}\mathsf{P}_U\right)^{\spadesuit}f\right\|_{L^q(\lambda_n)}\leq C\left\|f\right\|_{L^p\left(B\left(0,\frac{1}{2}\right)\right)}, \tag{1.8}$$

in which we have replaced $\mathbf{1}_{U_0}\left(\mathcal{A}_{\mathbf{a}}\right)^{\spadesuit}f$ by the truncation $\left(\mathcal{A}_{\mathbf{a}}\mathsf{P}_U\right)^{\spadesuit}f=\sum_{I\in\mathcal{G}[U]}a_I\triangle^{\eta}_{I;\kappa}f$. This latter inequality is what we will prove in the remainder of this paper.

Lemma 1. *The probabilistic Fourier extension inequality (1.7) is implied by the truncated probabilistic extension inequality (1.8).*

The proof of Lemma 1, given at the end of the next subsection on main results, also gives the following lemma upon removing the expectations $\mathbb{E}^{\mu}_{2^{\mathcal{G}}}$ and the random coefficients a_I from the proof.

Lemma 2. *The* deterministic *Fourier extension inequality (1.2) is implied by the truncated deterministic inequality,*

$$\left\|T\sum_{I\in\mathcal{G}[U]}\triangle^{\eta}_{I;\kappa}f\right\|_{L^q(\lambda_n)}\leq C\left\|f\right\|_{L^p\left(B\left(0,\frac{1}{2}\right)\right)}. \tag{1.9}$$

1.2. **The main results and a brief history.** The following Fourier extension conjecture arose from unpublished work of E. Stein in 1967, see e.g. [Ste2, see the Notes at the end of Chapter IX, p. 432, where Stein proved the restriction conjecture for $1\leq p<\frac{4n}{3n+1}$] and [Ste],

$$\left(\int_{\mathbb{R}^n}\left|\mathcal{F}\left(f\sigma_{n-1}\right)\right|^p d\xi\right)^{\frac{1}{p}}\leq C\left(\int_{\mathbb{S}^{n-1}}\left|f\left(x\right)\right|^p d\sigma_{n-1}\left(x\right)\right)^{\frac{1}{p}},\qquad \text{for } \frac{2n}{n-1}<p\leq\infty. \tag{1.10}$$

Our probabilistic analogue of (1.10) is the following conjecture for the case $p=q$, where $\left(\mathcal{A}_{\mathbf{a}}\right)^{\spadesuit}=S_{\kappa,\eta}\mathcal{A}_{\mathbf{a}}\left(S_{\kappa,\eta}\right)^{-1}$ is the conjugation of the martingale transform $\mathcal{A}_{\mathbf{a}}$ with the bounded invertible linear map $S_{\kappa,\eta}$ used in constructing the smooth Alpert wavelets in Theorem 6 below.

Conjecture 3. *For $\kappa>\frac{n}{2}$*[1] *and notation as above,*

$$\mathbb{E}^{\mu}_{2^{\mathcal{G}}}\left\|T\mathbf{1}_{U_0}\left(\mathcal{A}_{\mathbf{a}}\right)^{\spadesuit}f\right\|_{L^p(\lambda_n)}\lesssim\left\|f\right\|_{L^p\left(B\left(0,\frac{1}{2}\right)\right)},\qquad \textit{if and only if } \frac{2n}{n-1}<p\leq\infty, \tag{1.11}$$

equivalently, the square function estimate,

$$\left\|\mathcal{S}_{T\mathbf{1}_{U_0}}f\right\|_{L^q(\lambda_n)}\lesssim\left\|f\right\|_{L^p\left(B\left(0,\frac{1}{2}\right)\right)},\qquad \textit{if and only if } \frac{2n}{n-1}<p\leq\infty, \tag{1.12}$$

where

$$\mathcal{S}_{T\mathbf{1}_{U_0}}f\equiv\left(\sum_{I\in\mathcal{G}[U]}\left|T\mathbf{1}_{U_0}\triangle^{n-1,\eta}_{I;\kappa}f\right|^2\right)^{\frac{1}{2}}.$$

Theorem 4 (Probabilistic extension conjecture)**.** *The probabilistic Fourier extension inequalities (1.11) and (1.12) hold in all dimensions $n\geq 2$.*

Here the implied constant in $\lesssim$ depends only on harmless quantities determined by context, which in the display (1.11) are n, p and U_0.

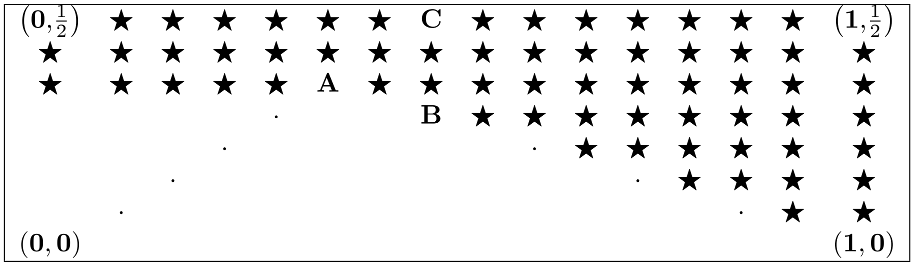
Sections 2 through 10 are devoted to proving Theorem 4. Some concluding remarks are made in Section 11.

Acknowledgement 5. *I am indebted to Hong Wang and Ruixiang Zhang for pointing out serious gaps in earlier versions of this paper, which claimed stronger results.*

[1]It seems likely this conjecture holds for the classical Haar expansion (it is of course implied by the Fourier extension conjecture), but we need $\kappa>\frac{n}{2}\geq 1$ in our proof of the smooth wavelet decomposition in Theorem 6.

There is a long history of progress on the Fourier extension conjecture in the past half century, and we refer the reader to the excellent survey articles by Thomas Wolff [Wol], Terence Tao [Tao] and Betsy Stovall [Sto] for this history up to 2019, as well as for connections with related conjectures and topics. Recently, a proof of the Kakeya set conjecture in $\mathbb{R}^3$ has been posted to the arXiv by Hong Wang and Joshua Zahl [WaZa]. See further references below.

The following $\left(\frac{1}{p},\frac{1}{q}\right)$-rectangle for boundedness of the extension operator illustrates this progression of positive results:



$$\mathbf{A}=\left(\frac{n-1}{2n},\frac{n-1}{2n}\right) \text{ and } \mathbf{B}=\left(\frac{1}{2},\frac{n-1}{2n+2}\right) \text{ and } \mathbf{C}=\left(\frac{1}{2},\frac{1}{2}\right)$$

The region marked with ★ is where boundedness of the extension operator (1.1) is known to fail, i.e. on and above the line $\frac{1}{q}=\frac{n-1}{2n}$, and strictly above the Knapp line joining $\mathbf{A}$ to $(1,0)$. The probabilistic analogue (1.4) also fails for these pairs $\left(\frac{1}{p},\frac{1}{q}\right)$, as is shown below. The point $\mathbf{B}$ on the Knapp line is the Stein-Tomas point, where boundedness is known from their 1975 result. Since the set of points $\left(\frac{1}{p},\frac{1}{q}\right)$ for which boundedness holds is both left-filled by embedding of L^p spaces on the sphere, and convex by interpolation, we see that as of 1975, the region consisting of the line joining $\mathbf{B}$ to $(1,0)$, and everything to the left of it, was known to be bounded for the extension operator. The point $\left(\frac{1}{2+\frac{4}{n}},\frac{1}{2+\frac{4}{n}}\right)$ was added by Tao [Tao4] in 2003, and points slightly better than $\left(\frac{1}{2+\frac{3}{n}},\frac{1}{2+\frac{3}{n}}\right)$ were added by Bourgain and Guth [BoGu, BoGu] in 2018.

Note also that any progress along the open diagonal line joining $(0,0)$ and $\mathbf{A}$, such as showing that $\left(\frac{1}{p},\frac{1}{p}\right)$ is bounded, yields boundedness for the convex hull of $\left(\frac{1}{p},\frac{1}{p}\right)$ and the line $\frac{1}{q}=0$, as well as all points to the left. Of course, even if the open diagonal segment joining $(0,0)$ and $\mathbf{A}$ was known to be bounded, this would still leave the open segment of the Knapp line joining $\mathbf{A}$ to $\mathbf{B}$.

Our probabilistic theorem shows that the boundedness region for the probabilistic extension conjecture includes all points not already eliminated for the extension conjecture, except possibly for the open segment of the Knapp line joining $\mathbf{A}$ to $\mathbf{B}$. Indeed, the conditions $q\geq p'\frac{n+1}{n-1}$ and $\frac{2n}{n-1}<q$ are necessary for the extension inequality (1.1) to hold, see e.g. [Tao]. The same arguments show that these conditions on p and q are necessary for the probabilistic analogue (1.4) to hold, upon considering individual smooth Alpert wavelets $h_{I;\kappa}^{\eta}$ (see below for definitions). Since σ_{n-1} is a finite measure, embedding and interpolation with the trivial $L^1\rightarrow L^\infty$ bound, together with Theorem 4, prove the probabilistic extension inequality for this range of exponents, except for the range $q=p'\frac{n+1}{n-1}$ and $1<p<\frac{2n}{n-1}$. Since the Stein Tomas result [Tom] captures the subcase of (1.1) when $1\leq p\leq 2$, this leaves only $q=p'\frac{n+1}{n-1}$ and $2<p<\frac{2n}{n-1}$ open in the probabilistic extension conjecture.

1.2.1. *Proof of reduction to the truncated inequality.* Here we prove Lemma 1.

Proof of Lemma 1. Using $f=\sum_{I\in\mathcal{G}}\triangle_{I;\kappa}^{n-1,\eta}f$ from the first line in (1.17) of Theorem 6 below, we write[2]

$$\mathbf{1}_{U_0}\left(\mathcal{A}_{\mathbf{a}}\right)^{\spadesuit} f=\mathbf{1}_{U_0}\sum_{I\in\mathcal{G}}a_I\,\triangle_{I;\kappa}^{n-1,\eta}f=\mathbf{1}_{U_0}\sum_{I\in\mathcal{G}[U]}a_I\,\triangle_{I;\kappa}^{n-1,\eta}f+\mathbf{1}_{U_0}\sum_{k=1}^{\infty}a_{\pi^{(k)}U_0}\,\triangle_{\pi^{(k)}U_0;\kappa}^{n-1,\eta}f\equiv L_1^{\mathbf{a}}f+L_2^{\mathbf{a}}f.$$

[2] I thank Cristian Rios for pointing out this simplification to an earlier proof.

since $\mathbf{1}_{U_0} \triangle_{I;\kappa}^{n-1,\eta} f$ vanishes if $I \notin \mathcal{G}[U] \cup \left\{\pi^{(k)} U_0\right\}_{k=1}^{\infty}$. Indeed, $\operatorname{Supp} \triangle_{I;\kappa}^{n-1,\eta} \subset (1+\eta) U$ which is disjoint from U_0 if $I \notin \mathcal{G}[U] \cup \left\{\pi^{(k)} U_0\right\}_{k=1}^{\infty}$. We will now show that

$$\begin{aligned} (1.13) \qquad \mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\| T L_1^{\mathbf{a}} f \right\|_{L^q} &= \mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\| T \mathbf{1}_{U_0} \sum_{I \in \mathcal{G}[U]} a_I \triangle_{I;\kappa}^{n-1,\eta} f \right\|_{L^q} \lesssim \mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\| T \sum_{I \in \mathcal{G}[U]} a_I \triangle_{I;\kappa}^{n-1,\eta} f \right\|_{L^q} , \\ \sup_{\mathbf{a}} \left\| T L_2^{\mathbf{a}} f \right\|_{L^q} &= \sup_{\mathbf{a}} \left\| T \mathbf{1}_{S_0} \sum_{k=1}^{\infty} a_{\pi^{(k)} U_0} \triangle_{\pi^{(k)} U_0;\kappa}^{n-1,\eta} f \right\|_{L^q} \lesssim \|f\|_{L^p\left(B\left(0,\frac{1}{2}\right)\right)} , \end{aligned}$$

which is easily seen to complete the proof that (1.8) implies (1.7).

To see the first line in (1.13), choose a rectangle R_0 in $\mathbb{R}^n$ with base U_0 and height 1 so that $R_0 \cap \mathbb{S}^{n-1} = \Phi(U_0)$. Then $\Phi_* \mathbf{1}_{U_0} = \mathbf{1}_{R_0} \Phi_*$, and since $\mathcal{F} \mathbf{1}_{R_0} \mathcal{F}^{-1}$ is a bounded Fourier multiplier on $L^q(\mathbb{R}^n)$ for all $1 < q < \infty$, we obtain

$$\begin{aligned} & \mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\| T L_1 f \right\|_{L^q} = \mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\| \mathcal{F} \Phi_* \mathbf{1}_{U_0} \sum_{I \in \mathcal{G}[U]} a_I \triangle_{I;\kappa}^{n-1,\eta} f \right\|_{L^q} \\ = \; & \mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\| \mathcal{F} \mathbf{1}_{R_0} \Phi_* \sum_{I \in \mathcal{G}[U]} a_I \triangle_{I;\kappa}^{n-1,\eta} f \right\|_{L^q} = \mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\| \left(\mathcal{F} \mathbf{1}_{R_0} \mathcal{F}^{-1} \right) \mathcal{F} \Phi_* \sum_{I \in \mathcal{G}[U]} a_I \triangle_{I;\kappa}^{n-1,\eta} f \right\|_{L^q} \\ \lesssim \; & \mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\| \mathcal{F} \Phi_* \sum_{I \in \mathcal{G}[U]} a_I \triangle_{I;\kappa}^{n-1,\eta} f \right\|_{L^q} = \mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\| T \sum_{I \in \mathcal{G}[U]} a_I \triangle_{I;\kappa}^{n-1,\eta} f \right\|_{L^q} . \end{aligned}$$

Now we turn to proving the second line in (1.13). Let ψ be a smooth bump function that is 1 on U_0 and supported in U. Then arguing once more as above,

$$\begin{aligned} & \left\| T \mathbf{1}_{U_0} L_2^{\mathbf{a}} f \right\|_{L^q} = \left\| \mathcal{F} \Phi_* \mathbf{1}_{U_0} \psi L_2^{\mathbf{a}} f \right\|_{L^q} = \left\| \mathcal{F} \mathbf{1}_{R_0} \Phi_* \psi L_2^{\mathbf{a}} f \right\|_{L^q} \\ = \; & \left\| \mathcal{F} \mathbf{1}_{R_0} \mathcal{F}^{-1} \mathcal{F} \Phi_* \psi L_2^{\mathbf{a}} f \right\|_{L^q} \lesssim \left\| \mathcal{F} \Phi_* \psi L_2^{\mathbf{a}} f \right\|_{L^q} = \left\| T \psi L_2^{\mathbf{a}} f \right\|_{L^q} , \end{aligned}$$

where

$$\psi L_2^{\mathbf{a}} f = \psi \sum_{k=1}^{\infty} a_{\pi^{(k)} U_0} \psi \triangle_{\pi^{(k)} U_0;\kappa}^{n-1,\eta} f = \sum_{k=1}^{\infty} a_{\pi^{(k)} U_0} \left\langle \mathbf{1}_{U_0} \left(S_{\kappa,\eta}\right)^{-1} f, h_{\pi^{(k)} U_0;\kappa} \right\rangle \psi h_{\pi^{(k)} U_0;\kappa}^{\eta} .$$

Thus we see that $\psi L_2^{\mathbf{a}} f$ is smooth and compactly supported upon using that (**i**) the functions $\psi h_{\pi^{(k)} U_0;\kappa}^{\eta}$ are smooth and compactly supported uniformly in k, and that (**ii**) we have the pointwise inquality,

$$\begin{aligned} & \left| \sum_{k=1}^{\infty} a_{\pi^{(k)} U_0} \left\langle \mathbf{1}_{U_0} \left(S_{\kappa,\eta}\right)^{-1} f, h_{\pi^{(k)} U_0;\kappa} \right\rangle \psi h_{\pi^{(k)} U_0;\kappa}^{\eta} \right| \lesssim \|\psi\|_{L^\infty} \sum_{k=1}^{\infty} \left\| \mathbf{1}_{U_0} \left(S_{\kappa,\eta}\right)^{-1} f \right\|_{L^1} \left\| h_{\pi^{(k)} U_0;\kappa} \right\|_{L^\infty}^2 \\ \lesssim \; & \sum_{k=1}^{\infty} \left\| \mathbf{1}_{U_0} \left(S_{\kappa,\eta}\right)^{-1} f \right\|_{L^p} \left\| h_{\pi^{(k)} U_0;\kappa} \right\|_{L^\infty}^2 \lesssim \sum_{k=1}^{\infty} \|f\|_{L^p} \frac{1}{\left|\pi^{(k)} U_0\right|} \lesssim \|f\|_{L^p} . \end{aligned}$$

Consequently, the Fourier transform $\widehat{\Phi_* \left(\psi L_2^{\mathbf{a}} f\right)}$ of the smooth surface measure $\Phi_* \left(\psi L_2^{\mathbf{a}} f\right)$ has decay

$$\left| \widehat{\Phi_* \left(\psi L_2^{\mathbf{a}} f\right)} (\xi) \right| \lesssim \|\psi\|_{C^{\frac{n}{2}+2}} \|f\|_{L^p} \left(1+|\xi|\right)^{-\frac{n-1}{2}} ,$$

by e.g. [Ste2, Theorem 1 page 348] or Theorem 28 below. Since this function is in $L^q(\mathbb{R}^n)$ for all $q > \frac{2n}{n-1}$, it follows that

$$\left\| T L_2^{\mathbf{a}} f \right\|_{L^q} \lesssim \|f\|_{L^p(U)} ,$$

which proves the second line in (1.13), and completes the proof that (1.8) implies (1.7). □

1.3. **Quick overview of the proof using smooth Alpert wavelets.** We begin with a short and informal narrative.

Narrative: In the theory of nonhomogeneous harmonic analysis, and especially that of two weight norm inequalities for the *Hilbert* transform, Nazarov, Treil and Volberg initiated the systematic use of weighted Haar wavelets to analyze boundedness. The Hilbert transform has kernel $\frac{1}{x-\xi}$, and thus the action of a Haar wavelet against such a kernel typically has geometric decay away from the origin, which permits 'error' off diagonal terms to be controlled. This two weight theory has concentrated mainly on the Hilbert space case $p=2$ in the past couple of decades, but more recently L^p estimates and square functions have attracted attention, especially with the recent work of Hytönen and Vuorinen.

At this point it becomes conceivable that square function and two weight techniques might be applicable to two weight L^p norm inequalities for the *Fourier* transform, such as the Fourier restriction conjecture, equivalent to the norm inequality with measures $d\sigma_{n-1}$ and $d\lambda_n$ in $\mathbb{R}^n$,

$$\left\Vert \mathcal{F}\left(f\sigma_{n-1}\right) \right\Vert_{L^p(\lambda_n)} \lesssim \left\Vert f\right\Vert_{L^p(\sigma_{n-1})} .$$

However, the kernel $K(x,\xi) = e^{-ix\cdot\xi}$ of the Fourier transform $\mathcal{F}$ is purely oscillatory with no decay at all, but this is partially offset by the curvature of the support of σ_{n-1}, that produces decay from the principle of stationary phase. Moreover, the action of a Haar wavelet against this kernel will be small if there is little variation of the kernel over the support of the wavelet (i.e. long wavelength), since the wavelet has vanishing mean, but this gain is limited by the absence of higher order vanishing moments in a Haar wavelet.

Addressing this defect, Alpert constructed wavelets with similar properties to those of Haar, but with additional vanishing moments that confer extra geometric gain. But even with Alpert wavelets in place of Haar wavelets, there is no geometric gain when the wavelength of the kernel is small compared to the size of the wavelet, due to the abrupt cutoffs in the dyadic construction of these wavelets.

In this paper we construct *smooth* Alpert wavelets that permit geometric decay when the wavelengths are small, i.e. when there is sufficient oscillation of the kernel over the support of the wavelet to permit gain from repeated integration by parts. Thus we will have gain except in the case of *resonance*, when there is neither sufficient smoothness nor oscillation in the restriction of the kernel to the support of either the $n-1$ or n dimensional wavelet. In these resonant situations, which form the core of difficulty in the deterministic Fourier extension conjecture, we must appeal to probability in order to obtain the desired L^4 bound needed for interpolation. The remainder of the paper holds without the intervention of probability.

Our proof of the probabilistic Fourier extension conjecture uses some techniques arising in the two weight testing theory of operator norms, [NTV4], [Vol], [LaSaShUr3], [SaShUr7], [AlSaUr] and [SaWi], that were in turn based on older work with roots in [FeSt], [DaJo], [Saw] and [Saw3], and followed by many other papers as well, such as [Hyt], [LaWi], [SaShUr12] and [HyVu] to mention just a few[3]. One of the main new ingredients used here is the construction of compactly supported smooth frames in L^p with derivative estimates adapted to the support, and as many vanishing moments as we wish. In fact, we will show that the wavelets $h_{I;\kappa}^{a,\eta}$ in the following theorem, can be constructed in the spirit of symbol smoothing, as appropriate convolutions of a certain approximate identity with the Alpert wavelets in [Alp], see also their weighted versions in [RaSaWi].

[3]Some of the deepest results in testing theory, namely the good/bad machinery of Nazarov, Treil and Volberg in e.g. [NTV4], the functional energy from [LaSaShUr3], the two weight inequalities for Poisson integrals from [Saw3], and the upside down corona and recursion from Lacey [Lac], are not used here. Some reasons for this are the lack of 'edge effects' in smooth Alpert wavelets, the lack of a paraproduct/stopping form decomposition, the 'niceness' of surface measure on the sphere and Lebesgue measure, and of course that the probabilistic conjecture is significantly weaker than the deterministic one. Indeed, the higher frequencies are damped to a greater extent by expectation, and this is why Kakeya phenomena do not enter into probabilistic arguments. On the other hand we make extensive use of pigeonholing into bilinear subforms according to the uncertainty principle, and then applying square function techniques for Alpert frames.

As already noted, for the proof of the probabilistic extension conjecture, it is enough to prove (1.8),

$$\mathbb{E}^{\mu}_{2^{\mathcal{G}}} \left\| T\left(\sum_{I\in\mathcal{G}[U]} a_I \bigtriangleup^{n-1,\eta}_{I;\kappa} f \right) \right\|_{L^p} \lesssim \|f\|_{L^p} \ .$$

However, we begin by writing the Fourier bilinear form $\left\langle T\left(\sum_{I\in\mathcal{G}[U]} a_I \bigtriangleup^{n-1,\eta}_{I;\kappa} f\right), g\right\rangle_{\mathbb{R}^n}$ as a finite sum of subforms

$$\mathsf{B}_{\mathcal{P}}(f,g) \equiv \sum_{(I,J)\in\mathcal{P}} \left\langle T\left(a_I \bigtriangleup^{n-1,\eta}_{I;\kappa} f\right), \bigtriangleup^{n,\eta}_{J;\kappa} g\right\rangle_{\mathbb{R}^n}$$

where $\mathcal{P}$ is a collection of pairs of dyadic cubes $I \in \mathcal{G}[U]$ and $J \in \mathcal{D}$, and where $\bigtriangleup^{n-1,\eta}_{I;\kappa}$ and $\bigtriangleup^{n,\eta}_{J;\kappa}$ are smooth Alpert pseudoprojections in $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$ respectively. This decomposition into subforms follows that used by Nazarov, Treil and Volberg in the setting of singular integrals with weighted Haar wavelets, but using the uncertainty principle to compare sizes of cubes here. There are six main subforms, the below $\mathsf{B}_{\text{below}}(f,g)$, above $\mathsf{B}_{\text{above}}(f,g)$, upper disjoint and distal $\mathsf{B}^{\text{upper}}_{\text{disjoint}}(f,g), \mathsf{B}^{\text{upper}}_{\text{distal}}(f,g)$, and lower disjoint and distal $\mathsf{B}^{\text{lower}}_{\text{disjoint}}(f,g), \mathsf{B}^{\text{lower}}_{\text{distal}}(f,g)$ subforms. The first two subforms are handled by the classical methods of integration by parts and stationary phase, but also use the smoothness and moment vanishing properties of the Alpert wavelets constructed in the next theorem, while the next two upper forms also use tangential integration by parts.

Finally, the last two most challenging forms, namely the lower disjoint and distal forms [4], are handled using properties of smooth Alpert wavelets with expectation taken over involutive smooth Alpert multipliers. While the deterministic *form* estimates for the previous four forms imply corresponding deterministic *norm* estimates by duality, this is no longer true for the probabilistic estimates we obtain, and it is important that we obtain the stronger probabilistic *norm* estimates in these cases. In fact, we will obtain L^2 and average L^4 norm estimates for smooth Alpert pseudoprojections (essentially because these spaces have the upper majorant property), which can then be interpolated to obtain the required norm bounds. However, this argument fails without expectation, and so fails to obtain the Fourier extension conjecture, whose attack requires far more sophisticated techniques. See Proposition 32, and Lemmas 33 and 34 below.

Here is the smooth compactly supported frame of wavelets for L^p that we will use[5].

Theorem 6. *Let $n,\kappa\in\mathbb{N}$ with $\kappa > \frac{n}{2}$, and $\eta > 0$ be sufficiently small depending on n and κ. Then there are a bounded invertible linear map $S_{\kappa,\eta} : L^p \to L^p$ $(1<p<\infty)$ satisfying*

$$\|\mathrm{Id} - S_{\kappa,\eta}\|_{L^p\to L^p} \leq C_{n,p}\eta \ , \tag{1.14}$$

and 'wavelets' $\left\{h^a_{I;\kappa}\right\}_{I\in\mathcal{D},\ a\in\Gamma_n}$ and $\left\{h^{a,\eta}_{I;\kappa}\right\}_{I\in\mathcal{D},\ a\in\Gamma_n}$ (with Γ_n a finite index set depending only on κ and n), and corresponding projections and pseudoprojections $\left\{\bigtriangleup_{I;\kappa}\right\}_{I\in\mathcal{D}}$ and $\left\{\bigtriangleup^{\eta}_{I;\kappa}\right\}_{I\in\mathcal{D}}$ defined by

$$\bigtriangleup_{I;\kappa} f \equiv \sum_{a\in\Gamma_n} \left\langle f, h^a_{I;\kappa}\right\rangle h^a_{I;\kappa} \text{ and } \bigtriangleup^{\eta}_{I;\kappa} f \equiv \sum_{a\in\Gamma_n} \left\langle \left(S_{\kappa,\eta}\right)^{-1} f, h^a_{I;\kappa}\right\rangle h^{a,\eta}_{I;\kappa} \ ,$$

satisfying

(1) *the standard properties,*

$$\begin{aligned} \left\|h^{a,\eta}_{I;\kappa}\right\|_{L^2} &\approx \left\|h^a_{I;\kappa}\right\|_{L^2} = 1, \\ \operatorname{Supp} h^a_{I;\kappa} &\subset I \text{ and } \operatorname{Supp} h^{a,\eta}_{I;\kappa} \subset (1+\eta) I, \\ \left\|\nabla^m h^{a,\eta}_{I;\kappa}\right\|_{\infty} &\leq C_m \left(\frac{1}{\eta\ell(I)}\right)^m \frac{1}{\sqrt{|I|}}, \qquad \text{for all } m\geq 0, \\ \int h^a_{I;\kappa}(x)\, x^{\alpha} dx &= \int h^{a,\eta}_{I;\kappa}(x)\, x^{\alpha} dx = 0, \qquad \text{for all } 0\leq|\alpha|<\kappa. \end{aligned} \tag{1.15}$$

[4]challenging because of the resonance that arises when the cubes I and J are appropriately positioned and sized, with the consequence that neither integration by parts nor moment vanishing can be put to use. In fact, it was precisely this difficulty that led to the serious gap in an earlier version v4 of this paper, and which was pointed out to the author by Hong Wang and Ruixiang Zhang.

[5]This particular theorem does not appear to be in the literature on frames.

(2) *and for each $a\in\Gamma_n$ the wavelets $h^a_{I;\kappa}$ and $h^{a,\eta}_{I;\kappa}$ are translations and L^2-dilations of the unit wavelets $h^a_{Q_0;\kappa}$ and $h^{a,\eta}_{Q_0;\kappa}$ respectively, where $Q_0=[0,1)^n$ is the unit cube in $\mathbb{R}^n$,*

$$h^a_{I;\kappa}=\sqrt{\frac{|Q_0|}{|I|}}h^a_{Q_0;\kappa}\circ\varphi_I \text{ and } h^{a,\eta}_{I;\kappa}=\sqrt{\frac{|Q_0|}{|I|}}h^{a,\eta}_{Q_0;\kappa}\circ\varphi_I\ , \tag{1.16}$$

where $\varphi_I:I\to Q_0$ is the affine map taking I one-to-one and onto Q_0,

(3) *and for all $1<p<\infty$,*

$$f=\sum_{I\in\mathcal{D},\ a\in\Gamma_n}\triangle^a_{I;\kappa}f=\sum_{I\in\mathcal{D},\ a\in\Gamma_n}\triangle^{a,\eta}_{I;\kappa}f,\qquad \text{with convergence in norm for } f\in L^p\cap L^2, \tag{1.17}$$

$$\left\|\left(\sum_{I\in\mathcal{D},\ a\in\Gamma_n}\left|\triangle^a_{I;\kappa}f\right|^2\right)^{\frac12}\right\|_{L^p(\mathbb{R}^n)}\approx\left\|\left(\sum_{I\in\mathcal{D},\ a\in\Gamma_n}\left|\triangle^{a,\eta}_{I;\kappa}f\right|^2\right)^{\frac12}\right\|_{L^p(\mathbb{R}^n)}\approx\|f\|_{L^p(\mathbb{R}^n)}\ ,\qquad \text{for } f\in L^p\cap L^2,$$

(4) *and for all $I\in\mathcal{D}$,*

$$h^a_{Q;\kappa}(x)=h^{a,\eta}_{Q;\kappa}(x)\ ,\qquad \text{for } x\in\mathbb{R}^n\setminus\mathcal{H}_\eta(Q)\ ,$$

where $\mathcal{H}_\eta(Q)$ is the η-halo of the skeleton of Q defined in (2.4) below.

(5) *and finally, the unsmoothed operators $\triangle_{I;\kappa}$ are self-adjoint orthogonal projections*[6],

$$\triangle_{I;\kappa}\triangle_{J;\kappa}=\begin{cases}\triangle_{I;\kappa} & \text{if}\quad I=J\\ 0 & \text{if}\quad I\neq J\end{cases}\ . \tag{1.18}$$

Remark 7. *This theorem shows that the collection of 'almost' L^2 projections $\left\{\triangle^{\eta,a}_{I;\kappa}\right\}_{I\in\mathcal{D},\ a\in\Gamma_n}$ is a 'frame' for the Banach space L^p, $1<p<\infty$. The case $\eta=0$ of (1.17) was obtained in the generality of doubling measures μ in* [SaWi].

Acknowledgement 8. *I thank Brett Wick for instigating our work on two weight L^p norm inequalities in* [LaWi]*, Michel Alexis and Ignacio Uriarte-Tuero for completing in our joint paper* [AlSaUr] *the work begun in* [Saw6] *on doubling measures, and Michel and Jose Luis Luna-Garcia for our work* [AlLuSa] *on L^p frames. Ideas from these papers have played a key role in the development of the arguments used here, as well as ideas from past collaborations and other works. I also thank Cristian Rios for valuable discussions and critical reading of portions of the manuscript, including a fruitful week long visit to Hamilton. Finally, I thank Ruixiang Zhang for many enlightening comments, and for pointing to several problems in the proof.*

1.3.1. *Organization of the paper.* In the next section we will construct and prove the required properties of smooth Alpert wavelets, and in Section 3 we introduce the extension operator and recall what we need regarding stationary phase. This material is well-known but we repeat it here due to the explicit error estimates we use. In Section 4 we discuss the initial wavelet decompositions into various subforms and describe the classical and well-known decay principles we use. Then in Section 5 we turn to the interpolation of L^2 and L^4 estimates using probability. Then in Sections 6, 7 and 8 we will control the below, above and upper disjoint forms respectively in the deterministic sense. Then in Section 9 we will use probability to control the lower disjoint form by averaging over smooth Alpert martingale transforms. Then we collect these results to finish the proof of the probabilistic Fourier extension theorem in Section 10, and in Section 11 we make some concluding comments.

1.4. **The initial setup.** Fix a small cube U_0 in $\mathbb{R}^{n-1}$ with side length a negative power of 2, and such that there is a translation $\mathcal{G}$ of the standard grid on $\mathbb{R}^{n-1}$ with the property that $U_0\in\mathcal{G}$, the grandparent $U\equiv\pi^{(2)}_{\mathcal{G}}U_0$ of U_0 has the origin as a vertex, and U_0 is an interior grandchild of U_0, so that

$$U_0,U\in\mathcal{G} \text{ with } U_0\subset\frac12U. \tag{1.19}$$

Now parameterize a patch of the sphere $\mathbb{S}^{n-1}$ in the usual way, i.e. $\Phi:U\to\mathbb{S}^{n-1}$ by

$$z=\Phi(x)\equiv\left(x,\sqrt{1-|x|^2}\right)=\left(x_1,x_2,...,x_{n-1},\sqrt{1-|x|^2}\right).$$

[6]The operators $\triangle^\eta_{I;\kappa}$ are neither self-adjoint, projections nor orthogonal, but come close as we will see.

For $f\in L^{p}\left(B_{n-1}\left(0,\frac{1}{2}\right)\right)$ define

$$Tf\left(\xi\right)\equiv\mathcal{F}\left(\Phi_{\ast}\left[f\left(x\right)dx\right]\right)=\int_{B_{n-1}\left(0,\frac{1}{2}\right)}e^{-i\Phi\left(x\right)\cdot\xi}f\left(x\right)\frac{dx}{\left\vert \det\Phi\left(x\right)\right\vert }, \tag{1.20}$$

where $\Phi_{\ast}\left[f\left(x\right)dx\right]$ is the pushforward of the measure $f\left(x\right)dx$ in $B_{n-1}\left(0,\frac{1}{2}\right)$ to the patch of sphere $\Phi\left(B_{n-1}\left(0,\frac{1}{2}\right)\right)$ lying above $B_{n-1}\left(0,\frac{1}{2}\right)$, and that we typically abuse notation by ignoring the harmless factor $\frac{1}{\left\vert \det\Phi\left(x\right)\right\vert }$. Recall that the Fourier extension inequality is equivalent to (1.2). The bilinear form associated to $T\mathbf{1}_{U_0}$ in (1.2) can be decomposed by,

$$\left\langle T\mathbf{1}_{U_0}f,g\right\rangle=\left\langle T\mathbf{1}_{U_0}\left(\sum_{I\in\mathcal{G}}\triangle_{I;\kappa}^{n-1}f\right),\sum_{J\in\mathcal{D}}\triangle_{J;\kappa}^{n}g\right\rangle=\sum_{\left(I,J\right)\in\mathcal{G}\times\mathcal{D}}\left\langle T\mathbf{1}_{U_0}\triangle_{I;\kappa}^{n-1}f,\triangle_{J;\kappa}^{n}g\right\rangle\ ,$$

where $\left\{\triangle_{J;\kappa}^{n}\right\}_{J\in\mathcal{D}}$ is an Alpert basis of projections for $L^{2}\left(\mathbb{R}^{n}\right)$, and $\left\{\triangle_{I;\kappa}^{n-1}\right\}_{I\in\mathcal{G}}$ is an Alpert basis of projections for $L^{2}\left(\mathbb{R}^{n-1}\right)$. Using rotation invariance, the Fourier extension conjecture is shown at the beginning of Section 3 below, to be equivalent to boundedness of $T\mathbf{1}_{U_0}$, taken over a finite collection of patches $\Phi\left(U_0\right)$.

Notation 9. *We are using the index $n-1$ or n in the superscript of the notation $\triangle_{I;\kappa}^{n-1,\eta}f$ for an Alpert projection, to denote whether the wavelet lives in $\mathbb{R}^{n-1}$ or in $\mathbb{R}^{n}$. The index η in the superscript denotes the smoothness injected by convolution in the construction of the smooth Alpert wavelets below. Moreover, we usually suppress the index $a\in\Gamma$ that runs over the set of all Alpert wavelets associated with a given cube.*

However, in order to carry out the standard two weight approach to bounding T, it will be necessary to fix $\kappa\in\mathbb{N}$, $\kappa>\frac{n}{2}$, and instead expand the bilinear form $\left\langle T\left(\mathsf{P}_{U}\right)^{\spadesuit}f,g\right\rangle=\left\langle T\sum_{I\in\mathcal{G}\left[U\right]}\triangle_{I;\kappa}^{n-1,\eta}f,g\right\rangle$, corresponding to the equivalent inequality (1.9), in terms of the *smooth* κ-Alpert decompositions of f and g,

$$\left\langle T\left(\mathsf{P}_{U}\right)^{\spadesuit}f,g\right\rangle=\sum_{\left(I,J\right)\in\mathcal{G}\left[U\right]\times\mathcal{D}}\left\langle T\triangle_{I;\kappa}^{n-1,\eta}f,\triangle_{J;\kappa}^{n,\eta}g\right\rangle,$$

so as to exploit the cancellation inherent in the oscillatory kernel $e^{-i\Phi\left(x\right)\cdot\xi}$ of the operator T_{S}.

Definition 10. *A subset E of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^{n}$ is said to be a* ball *if it is the intersection of the sphere with a halfspace, and is said to be a* pseudoball with constant C_{pseudo}*, if there are concentric balls B_1 and B_2 such that*

$$B_1\subset E\subset B_2\ \text{and}\ \left\vert B_2\right\vert\leq C_{\mathrm{pseudo}}\left\vert B_1\right\vert, \tag{1.21}$$

where $\left\vert E\right\vert$ denotes surface measure on the sphere. We simply say that E is a pseudoball *when C_{pseudo} is understood from context, and we will sometimes define a 'center' of E to be the center (not uniquely determined) of the balls B_1 and B_2 in (1.21).*

Definition 11. *Given a subset F of Euclidean space $\mathbb{R}^{n}$, we define the tangential and radial 'projections' of F, onto $\mathbb{S}^{n-1}$ and $[0,\infty)$ respectively, by*

$$\pi_{\tan}\left(F\right)\equiv\left\{\frac{\xi}{\left\vert\xi\right\vert}:\xi\in F\right\}\ \text{and}\ \pi_{\mathrm{rad}}\left(F\right)\equiv\left\{\left\vert\xi\right\vert:\xi\in F\right\}.$$

Then for C_{pseudo} chosen large enough in (1.21), the subsets $\Phi\left(I\right)$ and $\pi_{\tan}\left(J\right)$ of the sphere $\mathbb{S}^{n-1}$ are pseudoballs with constant C_{pseudo}, for all $I\in\mathcal{G}\left[U\right]$ and $J\in\mathcal{D}$. For $E\subset\mathbb{S}^{n-1}$, we denote by $-E$ the set antipodal to E, i.e. $-E=\left\{\zeta\in\mathbb{S}^{n-1}:-\zeta\in E\right\}$.

We now divide the collection of pairs $\left(I,J\right)\in\mathcal{G}\left[U\right]\times\mathcal{D}$ according to the relative size and location of their associated pseudoballs $\Phi\left(I\right)$ and $\pi_{\tan}\left(J\right)$, as dictated by the uncertainty principle:

$$\mathcal{G}\left[U\right]\times\mathcal{D}\subset\mathcal{P}\ \cup\ \mathcal{P}^{-}\ , \tag{1.22}$$

$$\begin{aligned}\text{where }\mathcal{P}&=\mathcal{P}_0\ \cup\ \bigcup_{m=1}^{\infty}\mathcal{P}_m\ \cup\ \mathcal{R}\cup\ \mathcal{X}\ ,\\ \text{and }\mathcal{P}^{-}&=\left\{\left(I,-J\right):\left(I,J\right)\in\mathcal{P}\right\}\ ,\end{aligned}$$

and where

$$\begin{aligned}
\mathcal{P}_0 &\equiv \{(I,J)\in\mathcal{G}[U]\times\mathcal{D}:\pi_{\tan}(J)\subset\Phi(C_{\mathrm{pseudo}}I)\} \ ,\\
\mathcal{P}_m &\equiv \left\{(I,J)\in\mathcal{G}[U]\times\mathcal{D}:2^{m+1}I\subset S \text{ and } \pi_{\tan}(J)\subset\Phi\left(2^{m+1}C_{\mathrm{pseudo}}I\right)\setminus\Phi\left(2^m\frac{1}{C_{\mathrm{pseudo}}}I\right)\right\}, \qquad 1\le m\le cs \ ,\\
\mathcal{R} &\equiv \{(I,J)\in\mathcal{G}[U]\times\mathcal{D}:\Phi(I)\subset\pi_{\tan}(C_{\mathrm{pseudo}}J)\} \ ,\\
\mathcal{X} &\equiv \left\{(I,J)\in\mathcal{G}[U]\times\mathcal{D}:J\subset\mathbb{R}^n_+ \text{ and } \pi_{\tan}(C_{\mathrm{pseudo}}J)\cap\Phi(2U)=\emptyset\right\} \ .
\end{aligned}$$

Note that there is some bounded overlap among the pairs in this decomposition, but this overcounting is inconsequential. Finally we point out that it suffices to show that

$$\left|\sum_{(I,J)\in\mathcal{P}}\left\langle T\,\triangle^{n-1,\eta}_{I;\kappa}f,\triangle^{n,\eta}_{J;\kappa}g\right\rangle\right|\lesssim\|f\|_{L^p}\|g\|_{L^{p'}} \ ,$$

since $(I,J)\in\mathcal{P}^-$ if and only if $(I,-J)\in\mathcal{P}$, and this amounts to replacing the kernel $e^{-i\Phi(x)\cdot\xi}$ with the kernel $e^{i\Phi(x)\cdot\xi}$, for which the estimates obtained below are identical.

2. Smooth Alpert frames in L^p spaces

Recall the Alpert projections $\{\triangle_{Q;\kappa}\}_{Q\in\mathcal{D}}$ and corresponding wavelets $\left\{h^a_{Q;\kappa}\right\}_{Q\in\mathcal{D},\ a\in\Gamma_n}$ of order κ in $\mathbb{R}^n$ that were constructed in B. Alpert [Alp] - see also [RaSaWi] for an extension to doubling measures, and for the terminology we use here. In fact, $\left\{h^a_{Q;\kappa}\right\}_{a\in\Gamma}$ is an orthonormal basis for the finite dimensional vector subspace of L^2 that consists of linear combinations of the indicators of the children $\mathfrak{C}(Q)$ of Q multiplied by polynomials of degree at most $\kappa-1$, and such that the linear combinations have vanishing moments on the cube Q up to order $\kappa-1$:

$$L^2_{Q;k}(\mu)\equiv\left\{f=\sum_{Q'\in\mathfrak{C}(Q)}\mathbf{1}_{Q'}p_{Q';k}(x):\int_Q f(x)\,x_i^{\ell}d\mu(x)=0,\quad\text{for } 0\le\ell\le k-1 \text{ and } 1\le i\le n\right\},$$

where $p_{Q';k}(x)=\sum_{\alpha\in\mathbb{Z}^n_+:|\alpha|\le k-1}a_{Q';\alpha}x^{\alpha}$ is a polynomial in $\mathbb{R}^n$ of degree $|\alpha|=\alpha_1+...+\alpha_n$ at most $\kappa-1$, and $x^{\alpha}=x_1^{\alpha_1}x_2^{\alpha_2}...x_{n-1}^{\alpha_{n-1}}$. Let $d_{Q;\kappa}\equiv\dim L^2_{Q;\kappa}(\mu)$ be the dimension of the finite dimensional linear space $L^2_{Q;\kappa}(\mu)$. Moreover, for each $a\in\Gamma_n$, we may assume the wavelet $h^a_{Q;\kappa}$ is a translation and dilation of the unit wavelet $h^a_{Q_0;\kappa}$, where $Q_0=[0,1)^n$ is the unit cube in $\mathbb{R}^n$.

2.1. **Alpert square functions.** It is shown in [SaWi, Corollary 14] (even for doubling measures in place of Lebesgue measure) that despite the failure of the κ-Alpert expansion to be a martingale when $\kappa\ge 2$, Burkholder's proof of the martingale transform theorem nevertheless carries over to prove, along with Khintchine's inequalities, that the L^p norm of the Alpert square function $\mathcal{S}f$ of f is comparable to the L^p norm of f, where

$$\mathcal{S}f(x)\equiv\left(\sum_{Q\in\mathcal{D},\ a\in\Gamma_n}\left|\triangle^a_{Q;\kappa}f(x)\right|^2\right)^{\frac12}, \qquad x\in\mathbb{R}^n.$$

Of course $\mathcal{S}f$ also depends on the grid $\mathcal{D}$ and κ, but we suppress this in the notation.

Theorem 12 (Sawyer and Wick [SaWi])**.** *For $\kappa\in\mathbb{N}$ and $1<p<\infty$, we have*

$$\|\mathcal{S}f\|_{L^p(\mathbb{R}^n)}\le C_{p,n,\kappa}\|f\|_{L^p(\mathbb{R}^n)}\ . \tag{2.1}$$

2.2. **Smoothing the Alpert wavelets.** Given a small positive constant $\eta>0$, define a smooth approximate identity by $\phi_\eta(x)\equiv\eta^{-n}\phi\left(\frac{x}{\eta}\right)$ where $\phi\in C^\infty_c(B_{\mathbb{R}^n}(0,1))$ has unit integral, $\int_{\mathbb{R}^n}\phi(x)\,dx=1$, and vanishing moments of *positive* order less than κ, i.e.

$$\int_{\mathbb{R}^n}\phi(x)\,x^{\gamma}dx=\delta^0_{|\gamma|}=\left\{\begin{array}{lll}1 & \text{if} & |\gamma|=0\\ 0 & \text{if} & 0<|\gamma|<\kappa\end{array}\right. \ . \tag{2.2}$$

In fact we may take for $\phi(x)$ a product function $\phi(x) = \prod_{i=1}^{n} \varphi(x_i)$ where $\varphi \in C_c^\infty((-1,1))$ satisfies

$$\int_{\mathbb{R}} \varphi(x)\, x^\gamma dx = \begin{cases} 1 & \text{if} \quad \gamma = 0 \\ 0 & \text{if} \quad 0 < \gamma < \kappa \end{cases}, \qquad \text{for } 1 \le i \le n. \tag{2.3}$$

One way to construct a function φ satisfying (2.3) is to pick $\chi \in C_c^\infty\left(\left(\frac{3}{4},1\right)\right)$ with $\int \chi(y)\, dy = 1$, a large $N \in \mathbb{N}$, and then for $\lambda \equiv (\lambda_1, ..., \lambda_N)$ to define,

$$\varphi_\lambda(x) = \sum_{m=1}^{N} \lambda_m \chi(2^m x).$$

Then with the change of variable $y = 2^m x$ we have,

$$\int \varphi_\lambda(x)\, x^\gamma dx = \sum_{m=1}^{N} \lambda_m \int \chi(2^m x)\, x^\gamma dx = \sum_{m=1}^{N} \lambda_m 2^{-m(\gamma+1)} \int \chi(y)\, y^\gamma dy = C_\gamma \sum_{m=1}^{N} \lambda_m 2^{-m(\gamma+1)}.$$

In order to achieve $\int \varphi_\lambda(x)\, x^\gamma dx = \begin{cases} 1 & \text{if} \quad \gamma = 0 \\ 0 & \text{if} \quad 0 < \gamma < \kappa \end{cases}$ we need to solve the linear system,

$$1 = \sum_{m=1}^{N} \lambda_m 2^{-m} \text{ and } 0 = \sum_{m=1}^{N} \lambda_m 2^{-m(\gamma+1)}, \qquad \text{for } 0 < \gamma < \kappa,$$

which in matrix form is

$$\mathbf{e}_1 = M_\kappa \boldsymbol{\lambda}. \qquad \text{where } M_\kappa \equiv \left[2^{-m\ell}\right]_{\substack{1 \le m \le N \\ 1 \le \ell \le \kappa}}.$$

We take $N \ge \kappa$ and observe that the square matrix $M_\kappa \equiv \left[2^{-m\ell}\right]_{\substack{1 \le m \le \kappa \\ 1 \le \ell \le \kappa}}$ has nonzero determinant, in fact $|\det M_\kappa|$ is bounded below by $2^{-\frac{\kappa^2(\kappa-1)}{2}}$. Indeed, the square Vandermonde matrix

$$V(x) = V(x_1, x_2, ..., x_n) \equiv \begin{bmatrix} x_1 & x_1^2 & \cdots & x_1^n \\ x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

has determinant $\det V(x) = \prod_{1 \le i < j \le n} (x_j - x_i)$. Thus with $x(\kappa) = \left(2^{-1}, 2^{-2}, ..., 2^{-\kappa}\right) \in \mathbb{R}^\kappa$, we have $V(x(\kappa)) = \left[2^{-m\ell}\right]_{\substack{1 \le m \le \kappa \\ 1 \le \ell \le \kappa}} = M_\kappa$ and so

$$|\det M_\kappa| = \prod_{1 \le i < j \le \kappa} \left|2^{-j} - 2^{-i}\right| \ge \prod_{1 \le i < j \le \kappa} 2^{-\kappa} = 2^{-\kappa \frac{\kappa(\kappa-1)}{2}}.$$

Thus we can find coefficients $\lambda \equiv (\lambda_1, ..., \lambda_N)$ such that $\varphi = \varphi_\lambda$ satisfies (2.3).

In the spirit of symbol smoothing for pseudodifferential operators, we define *smooth* Alpert 'wavelets' by

$$h_{Q;\kappa}^{a,\eta} \equiv h_{Q;\kappa}^{a} * \phi_{\eta \ell(Q)},$$

and we claim that $h_{Q;\kappa}^{a}$ and $h_{Q;\kappa}^{a,\eta}$ coincide away from the η-neighbourhood (often referred to as a 'halo')

$$\mathcal{H}_\eta(Q) \equiv \{x \in \mathbb{R}^n : \text{dist}(x, S_Q) < \eta\}, \tag{2.4}$$

of the skeleton $S_Q \equiv \bigcup_{Q' \in \mathfrak{C}_{\mathcal{D}}(Q)} \partial Q'$. Note that away from the skeleton, the Alpert wavelet $h_{Q;\kappa}^{a}$ restricts to a polynomial of degree less than κ on each dyadic child of Q. We now show the same for smooth Alpert wavelets away from the halo of the skeleton.

Lemma 13. *With notation as above and ϕ satisfying (2.2), we have*

$$h_{Q;\kappa}^{a}(x) = h_{Q;\kappa}^{a,\eta}(x), \qquad x \in \mathbb{R}^n \setminus \mathcal{H}_\eta(Q). \tag{2.5}$$

Proof. If $m_\alpha(x) \equiv x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$ is a multinomial, then

$$(m_\alpha * \phi)(x) = \sum_{0 \le \beta \le \alpha} \left(c_{\alpha,\beta} \int y^{\alpha-\beta} \phi(y)\, dy\right) x^\beta = x^\alpha = m_\alpha(x),$$

which shows that (2.5) holds. □

We also observe that for $0 \leq |\beta| < \kappa$,

$$\begin{aligned}
&\int h_{Q;\kappa}^{a,\eta}(x)\, x^{\beta} dx = \int \phi_{\eta\ell(I)} * h_{Q;\kappa}^{a}(x)\, x^{\beta} dx = \int\int \phi_{\eta\ell(I)}(y)\, h_{Q;\kappa}^{a}(x-y)\, x^{\beta} dx \\
= &\int \phi_{\eta\ell(I)}(y) \left\{ \int h_{Q;\kappa}^{a}(x-y)\, x^{\beta} dx \right\} dy = \int \phi_{\eta\ell(I)}(y) \left\{ \int h_{Q;\kappa}^{a}(x)\, (x+y)^{\beta}\, dx \right\} dy \\
= &\int \phi_{\eta\ell(I)}(y) \{0\}\, dy = 0,
\end{aligned}$$

by translation invariance of Lebesgue measure.

2.3. **The reproducing formula.** For the purposes of this subsection we will change notation from that in Theorem 6 in the introduction by defining

$$\triangle_{I;\kappa}^{\eta} f \equiv \sum_{a\in\Gamma_n} \left\langle f, h_{I;\kappa}^{a} \right\rangle h_{I;\kappa}^{a,\eta} = \left(\triangle_{I;\kappa} f\right) * \phi_{\eta\ell(I)} .$$

Next, for any grid $\mathcal{D}$, we wish to show that for $\eta > 0$ sufficiently small, the linear map $S_{\kappa,\eta}^{\mathcal{D}}$ defined by

$$S_{\kappa,\eta}^{\mathcal{D}} f \equiv \sum_{I\in\mathcal{D},\ a\in\Gamma_n} \left\langle f, h_{I;\kappa}^{a} \right\rangle h_{I;\kappa}^{a,\eta} = \sum_{I\in\mathcal{D}} \triangle_{I;\kappa}^{\eta} f , \qquad f \in L^p, \tag{2.6}$$

is bounded and invertible on L^p, and that we have the reproducing formula,

$$f(x) = \sum_{I\in\mathcal{D},\ a\in\Gamma_n} \left\langle \left(S_{\kappa,\eta}^{\mathcal{D}}\right)^{-1} f, h_{I;\kappa}^{a} \right\rangle\ h_{I;\kappa}^{a,\eta}(x), \qquad \text{for all } f \in L^p \cap L^2,$$

with convergence in the L^p norm. Since $\kappa > \frac{n}{2}$ is fixed throughout our arguments we will often write $S_{\eta}^{\mathcal{D}}$ instead of $S_{\kappa,\eta}^{\mathcal{D}}$ in the sequel.

Proof of Theorem 6. Theorem 6 follows easily, together with what was proved just above, from Theorem 14 below if we define the pseudoprojection $\triangle_{I;\kappa}^{\eta}$ in Theorem 6 as the pseudoprojection $\widetilde{\triangle}_{I;\kappa}^{\eta}$ in Theorem 14. □

We include arbitrary grids $\mathcal{D}$ in Theorem 14 since this may be useful in other contexts where probability of grids plays a role, originating with the work of Nazarov, Treil and Volberg, see e.g. [NTV4] and [Vol], and references given there.

Theorem 14. *Let $n \geq 2$ and $\kappa \in \mathbb{N}$ with $\kappa > \frac{n}{2}$. Then there is $\eta_0 > 0$ depending on n and κ such that for all $0 < \eta < \eta_0$, and for all grids $\mathcal{D}$ in $\mathbb{R}^n$, and all $1 < p < \infty$, there is a bounded invertible operator $S_{\eta}^{\mathcal{D}} = S_{\kappa,\eta}^{\mathcal{D}}$ on L^p, and a positive constant $C_{p,n,\eta}$ such that the collection of functions $\left\{ h_{I;\kappa}^{a,\eta} \right\}_{I\in\mathcal{D},\ a\in\Gamma_n}$ is a $C_{p,n,\eta}$-frame for L^p, by which we mean*[7],

$$\begin{aligned}
f(x) &= \sum_{I\in\mathcal{D},\ a\in\Gamma_n} \widetilde{\triangle}_{I;\kappa}^{\eta} f(x), \qquad \textit{for a.e. } x \in \mathbb{R}^n, \textit{ and for all } f \in L^p, \\
\textit{where } \widetilde{\triangle}_{I;\kappa}^{\eta} f &\equiv \sum_{a\in\Gamma_n} \left\langle \left(S_{\eta}^{\mathcal{D}}\right)^{-1} f, h_{I;\kappa}^{a} \right\rangle\ h_{I;\kappa}^{a,\eta} ,
\end{aligned} \tag{2.7}$$

and with convergence of the sum in the L^p norm, and

$$\frac{1}{C_{p,n,\eta}} \|f\|_{L^p} \leq \left\| \left(\sum_{I\in\mathcal{D}} \left| \widetilde{\triangle}_{I;\kappa}^{\eta} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p}, \left\| \left(\sum_{I\in\mathcal{D}} \left| \triangle_{I;\kappa}^{\eta} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \leq C_{p,n,\eta} \|f\|_{L^p} ,$$
$$\textit{for all } f \in L^p.$$

Notation 15. *We will often drop the index a parameterized by the finite set Γ_n as it plays no essential role in most of what follows, and it will be understood that when we write*

$$\triangle_{Q;\kappa}^{\eta} f = \left\langle f, h_{Q;\kappa} \right\rangle h_{Q;\kappa}^{\eta},$$

[7]See [AlLuSa] and [CaHaLa] for more detail on frames in L^p spaces.

we actually *mean the Alpert* pseudoprojection,

$$\triangle^{\eta}_{Q;\kappa} f = \sum_{a\in\Gamma_n} \left\langle f, h^a_{Q;\kappa} \right\rangle h^{a;\eta}_{Q;\kappa} \ .$$

Now we turn to two propositions that we will use in the proof of Theorem 14.

Proposition 16. *For $\kappa > \frac{n}{2}$ and $\eta > 0$ sufficiently small, we have*

$$\left\| S^{\mathcal{D}}_{\eta} f \right\|_{L^p} \approx \left\| f \right\|_{L^p} \ , \qquad \textit{for } f \in L^p \cap L^2 \textit{ and } 1 < p < \infty.$$

Proposition 17. *For $\kappa > \frac{n}{2}$ and $\eta > 0$ sufficiently small, we have*

$$\left\| \left(S^{\mathcal{D}}_{\eta} \right)^* f \right\|_{L^p} \approx \left\| f \right\|_{L^p} \ , \qquad \textit{for } f \in L^p \cap L^2 \textit{ and } 1 < p < \infty.$$

To prove these propositions, we will need some estimates on the inner products $\left\langle h^{\eta}_{I;\kappa}, h_{Q;\kappa} \right\rangle$ where one wavelet is smooth and the other is not. Fix a dyadic grid $\mathcal{D}$. We say that dyadic cubes Q_1 and Q_2 are *siblings* if $\ell(Q_1) = \ell(Q_2)$, $Q_1 \cap Q_2 = \emptyset$ and $\overline{Q_1} \cap \overline{Q_2} \neq \emptyset$, and we say they are *dyadic* siblings if in addition they have a common dyadic parent, i.e. $\pi_{\mathcal{D}} Q_1 = \pi_{\mathcal{D}} Q_2$. Finally, we define $\mathrm{Car}(Q)$ to be the set of $I \in \mathcal{D}$ with $\ell(I) < \ell(Q)$ such that I and Q share a face. We refer to these cubes I as Carleson cubes of Q, and note they can be either outside Q or inside Q. Finally, we may assume without loss of generality that η is a negative integer power of 2.

Lemma 18. *Suppose $\kappa \in \mathbb{N}$ with $\kappa > \frac{n}{2}$, $0 < \eta = 2^{-k} < 1$, and $I, Q \in \mathcal{D}$, where $\mathcal{D}$ is a grid in $\mathbb{R}^n$. Then we have*

$$\begin{aligned}
\left| \left\langle h^{\eta}_{Q;\kappa}, h_{Q;\kappa} \right\rangle \right| &\approx 1 \textit{ and } \left| \left\langle h^{\eta}_{Q;\kappa}, h_{Q';\kappa} \right\rangle \right| \lesssim \eta, && \textit{for } Q \textit{ and } Q' \textit{ siblings}, \\
\left| \left\langle h^{\eta}_{I;\kappa}, h_{Q;\kappa} \right\rangle \right| &\lesssim \eta \left(\frac{\ell(I)}{\ell(Q)} \right)^{\frac{n}{2}}, && \textit{for } I \in \mathrm{Car}(Q), \\
\left| \left\langle h^{\eta}_{I;\kappa}, h_{Q;\kappa} \right\rangle \right| &\lesssim \eta \left(\frac{\ell(Q)}{\ell(I)} \right)^{\frac{n}{2}-1}, && \textit{for } Q \in \mathrm{Car}(I) \textit{ and } \ell(Q) \geq \eta \ell(I), \\
\left| \left\langle h^{\eta}_{I;\kappa}, h_{Q;\kappa} \right\rangle \right| &\lesssim \frac{1}{\eta^{\kappa}} \left(\frac{\ell(Q)}{\ell(I)} \right)^{\kappa+\frac{n}{2}}, && \textit{for } \ell(Q) \leq \eta \ell(I) \textit{ and } Q \cap \mathcal{H}_{\frac{\eta}{2}}(I) \neq \emptyset, \\
\left\langle h^{\eta}_{I;\kappa}, h_{Q;\kappa} \right\rangle &= 0, && \textit{in all other cases.}
\end{aligned}$$

Proof. Fix a grid $\mathcal{D}$, and take $0 < \eta < 1$. We have

$$\left\langle h^{\eta}_{Q;\kappa}, h_{Q;\kappa} \right\rangle = \left\langle h_{Q;\kappa}, h_{Q;\kappa} \right\rangle + \left\langle h^{\eta}_{Q;\kappa} - h_{Q;\kappa}, h_{Q;\kappa} \right\rangle = 1 + \int_{\mathcal{H}_{\eta}(Q)} \left(h^{\eta}_{Q;\kappa} - h_{Q;\kappa} \right)(x)\, h_{Q;\kappa}(x)\, dx,$$

where

$$\left| \int_{\mathcal{H}_{\eta}(Q)} \left(h^{\eta}_{Q;\kappa} - h_{Q;\kappa} \right)(x)\, h_{Q;\kappa}(x)\, dx \right| \lesssim \left\| h^{\eta}_{Q;\kappa} - h_{Q;\kappa} \right\|_{\infty} \left\| h_{Q;\kappa} \right\|_{\infty} \left| \mathcal{H}_{\eta}(Q) \right| \lesssim \frac{1}{\sqrt{|Q|}} \frac{1}{\sqrt{|Q|}} \eta \left| Q \right| = \eta.$$

Next we note that if I is a dyadic cube and $Q\in\operatorname{Car}(I)$, then $Q\cap\mathcal{H}_{\eta}(I)\neq\emptyset$ and $\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}\right\rangle\neq 0$ where $\eta=2^{-k}$ imply that $\operatorname{Supp}h_{Q;\kappa}=Q\subset\mathcal{H}_{\eta}(I)$. If $Q\subset\mathcal{H}_{\eta}(I)$, then we have

$$\begin{aligned}
&\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}\right\rangle=\int_{\mathcal{H}_{\eta}(I)}\mathbf{1}_{Q}h_{I;\kappa}^{\eta}(x)\,h_{Q;\kappa}(x)\,dx=\int_{Q\cap\mathcal{H}_{\eta}(I)}\left(h_{I;\kappa}\ast\phi_{\eta\ell(I)}\right)(x)\,h_{Q;\kappa}(x)\,dx\\
=&\int_{Q\cap\mathcal{H}_{\eta}(I)}\left\{\int_{I}h_{I;\kappa}(y)\,\phi_{\eta\ell(I)}(x-y)\,dy\right\}h_{Q;\kappa}(x)\,dx=\int_{I}h_{I;\kappa}(y)\left\{\int_{Q\cap\mathcal{H}_{\eta}(I)}\phi_{\eta\ell(I)}(x-y)\,h_{Q;\kappa}(x)\,dx\right\}dy\\
=&\int_{I\cap 2\eta\ell(I)Q}h_{I;\kappa}(y)\left\{\int_{Q\cap\mathcal{H}_{\eta}(I)}\left[\phi_{\eta\ell(I)}(x-y)-\sum_{j=0}^{\kappa-1}\left((x-c_{Q})\cdot\nabla\right)^{j}\phi_{\eta\ell(I)}(c_{Q}-y)\right]h_{Q;\kappa}(x)\,dx\right\}dy\\
\leq&\left\|h_{I;\kappa}\right\|_{\infty}\left\|\left(\nabla^{\kappa}\phi_{\eta\ell(I)}\right)\right\|_{\infty}\ell(Q)^{\kappa}\left\|h_{Q;\kappa}\right\|_{\infty}\int_{B(c_{Q},\eta\ell(I))}\int_{Q\cap\mathcal{H}_{\eta}(I)}dxdy\\
\lesssim&\sqrt{\frac{1}{|I|}}\left\|\nabla^{\kappa}\phi\right\|_{\infty}\left(\frac{1}{\eta\ell(I)}\right)^{n+\kappa}\ell(Q)^{\kappa}\sqrt{\frac{1}{|Q|}}\left|B(c_{Q},\eta\ell(I))\right|\left|Q\cap\mathcal{H}_{\eta}(I)\right|\lesssim\frac{1}{\eta^{\kappa}}\left(\frac{\ell(Q)}{\ell(I)}\right)^{\kappa+\frac{n}{2}},
\end{aligned}$$

since $\left\|h_{I;\kappa}\right\|_{\infty}\lesssim\sqrt{\frac{1}{|I|}}$, $\left\|h_{Q;\kappa}\right\|_{\infty}\lesssim\sqrt{\frac{1}{|Q|}}$ and $\left\|\nabla^{\kappa}\phi_{\eta\ell(I)}\right\|_{\infty}\leq\left\|\nabla^{\kappa}\phi\right\|_{\infty}\left(\frac{1}{\eta\ell(I)}\right)^{\kappa}$.

If $Q\in\operatorname{Car}(I)$ and $\ell(Q)\geq\eta\ell(I)$, then we have the trivial estimate

$$\left|\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}\right\rangle\right|\lesssim\eta\ell(I)\,\ell(Q)^{n-1}\sqrt{\frac{1}{|I|\,|Q|}}=\eta\left(\frac{\ell(Q)}{\ell(I)}\right)^{\frac{n}{2}-1}.$$

On the other hand, if $I\in\operatorname{Car}(Q)$, we claim that

$$\left|\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}\right\rangle\right|\lesssim\eta\left(\frac{\ell(I)}{\ell(Q)}\right)^{\frac{n}{2}}.$$

Indeed, this is clear if $Q\cap I=\emptyset$ since then $\left|\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}\right\rangle\right|\leq\eta\,|I|\sqrt{\frac{1}{|I|}}\sqrt{\frac{1}{|Q|}}$, while if $Q'\in\mathfrak{C}_{D}(I)$ is the child containing I, and if $\varphi(x-c_{Q'})$ is the polynomial whose restriction to Q' is $\left(\mathbf{1}_{Q'}h_{Q;\kappa}\right)(x)$, then $\left\langle h_{I;\kappa}^{\eta},\varphi\right\rangle=0$ and so

$$\left|\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}\right\rangle\right|=\left|\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}-\varphi\right\rangle\right|\lesssim\eta\sqrt{\frac{|I|}{|Q|}}=\eta\left(\frac{\ell(I)}{\ell(Q)}\right)^{\frac{n}{2}}.$$

□

We will also need the following consequence of the Marcinkiewicz interpolation theorem.

Lemma 19. *For $1<p<\infty$ and $\kappa\in\mathbb{N}$, we have*

$$\left\|\left(\sum_{I\in\mathcal{D}}\left(\frac{\left|\langle f,h_{I;\kappa}\rangle\right|}{|I|^{\frac{1}{2}}}\mathbf{1}_{\mathcal{H}_{\eta}(I)}(x)\right)^{2}\right)^{\frac{1}{2}}\right\|_{L^{p}}\leq C_{p,n}\eta^{\gamma_{p}}\left\|f\right\|_{L^{p}},$$

$$\text{where }\gamma_{p}\equiv\left\{\begin{array}{ccc}\frac{1}{2(p-1)} & \text{if} & p>2\\ \frac{1}{2} & \text{if} & p=2\\ \frac{p-1}{p(3-p)} & \text{if} & 1<p<2\end{array}\right..$$

Proof. Define the square function $\mathcal{R}_{\eta}$ by

$$\mathcal{R}_{\eta}f(x)\equiv\left(\sum_{I\in\mathcal{D}}\left(\frac{\left|\langle f,h_{I;\kappa}\rangle\right|}{|I|^{\frac{1}{2}}}\mathbf{1}_{I\cap\mathcal{H}_{\eta}(I)}(x)\right)^{2}\right)^{\frac{1}{2}}.$$

Using $\mathbf{1}_{\mathcal{H}_\eta(I)}(x) \lesssim M\mathbf{1}_{I\cap\mathcal{H}_\eta(I)}(x)$, the Fefferman-Stein vector valued maximal inequality [FeSt] yields,

$$\left\| \left(\sum_{I\in\mathcal{D}} \left(\frac{|\langle f, h_{I;\kappa}\rangle|}{|I|^{\frac{1}{2}}} \mathbf{1}_{\mathcal{H}_\eta(I)}(x) \right)^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \left\| \left(\sum_{I\in\mathcal{D}} \left(\frac{|\langle f, h_{I;\kappa}\rangle|}{|I|^{\frac{1}{2}}} M\mathbf{1}_{I\cap\mathcal{H}_\eta(I)}(x) \right)^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$
$$\lesssim \left\| \left(\sum_{I\in\mathcal{D}} \left(\frac{|\langle f, h_{I;\kappa}\rangle|}{|I|^{\frac{1}{2}}} \mathbf{1}_{I\cap\mathcal{H}_\eta(I)}(x) \right)^2 \right)^{\frac{1}{2}} \right\|_{L^p} = \|\mathcal{R}_\eta f(x)\|_{L^p} .$$

Now we note that

$$\|\mathcal{R}_\eta f\|_{L^p} \lesssim \left\| \left(\sum_{I\in\mathcal{D}} \left(\frac{|\langle f, h_{I;\kappa}\rangle|}{|I|^{\frac{1}{2}}} \mathbf{1}_I \right)^2 \right)^{\frac{1}{2}} \right\|_{L^p} = \left\| \left(\sum_{I\in\mathcal{D}} (\triangle_{I;\kappa} f)^2 \right)^{\frac{1}{2}} \right\|_{L^p} = \|\mathcal{R} f\|_{L^p} \approx \|f\|_{L^p}$$

and

$$\begin{aligned}
\|\mathcal{R}_\eta f\|_{L^2}^2 &= \int \sum_{I\in\mathcal{D}} \left(\frac{|\langle f, h_{I;\kappa}\rangle|}{|I|^{\frac{1}{2}}} \mathbf{1}_{I\cap\mathcal{H}_\eta(I)}(x) \right)^2 dx = \int \sum_{I,I'\in\mathcal{D}} \frac{|\langle f, h_{I;\kappa}\rangle|}{|I|^{\frac{1}{2}}} \frac{|\langle f, h_{I';\kappa}\rangle|}{|I'|^{\frac{1}{2}}} \mathbf{1}_{I\cap\mathcal{H}_\eta(I)}(x)\, \mathbf{1}_{I'\cap\mathcal{H}_\eta(I')}(x)\, dx \\
&= \sum_{I,I'\in\mathcal{D}} \frac{|\langle f, h_{I;\kappa}\rangle|}{|I|^{\frac{1}{2}}} \frac{|\langle f, h_{I';\kappa}\rangle|}{|I'|^{\frac{1}{2}}} |I\cap\mathcal{H}_\eta(I)\cap I'\cap\mathcal{H}_\eta(I')| \leq \sum_{I,I'\in\mathcal{D}} \frac{|\langle f, h_{I;\kappa}\rangle|}{|I|^{\frac{1}{2}}} \frac{|\langle f, h_{I';\kappa}\rangle|}{|I'|^{\frac{1}{2}}} \eta\, |I\cap I'| \\
&= \eta \int \sum_{I\in\mathcal{D}} \left(\frac{|\langle f, h_{I;\kappa}\rangle|}{|I|^{\frac{1}{2}}} \mathbf{1}_I(x) \right)^2 dx = \eta \int \sum_{I\in\mathcal{D}} \frac{|\langle f, h_{I;\kappa}\rangle|^2}{|I|} \mathbf{1}_I(x)\, dx = \eta \sum_{I\in\mathcal{D}} |\langle f, h_{I;\kappa}\rangle|^2 = \eta\, \|f\|_{L^2}^2 .
\end{aligned}$$

Thus the (linearizable) sublinear operator $\mathcal{R}_\eta$ maps $L^2 \to L^2$ with bound $B_2 \equiv \eta^{\frac{1}{2}}$, and maps $L^q \to L^q$ with bound $B_q \equiv C'_{n,q}$ for $1 < q < \infty$ and $q \neq 2$.

In the case $p > 2$, let $q = 2p$. Then by the scaled Marcinkiewicz theorem applied to $\mathcal{R}_\eta$ with exponents 2 and $q = 2p$, see e.g. [Tao2, Remark 29], we have

$$\|\mathcal{R}_\eta f\|_{L^p} \leq C''_{n,p} B_2^{1-\theta} B_{2p}^{\theta} = C''_{n,p} \eta^{\frac{1}{2}(1-\theta)} \left(C'_{n,2p}\right)^{\theta} = C_{n,p} \eta^{\frac{1}{2(p-1)}},$$

with $C_{n,p} = C''_{n,p} \left(C'_{n,2p}\right)^{\frac{p-2}{p-1}}$, since $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{2p}$ implies $1-\theta = \frac{1}{p-1}$.

In the case $1 < p < 2$, take $q = \frac{1+p}{2}$ and apply the scaled Marcinkiewicz theorem to $\mathcal{R}_\eta$ with exponents 2 and $q = \frac{1+p}{2}$ to obtain

$$\|\mathcal{R}_\eta f\|_{L^p} \leq C''_{n,p} B_2^{1-\theta} B_{\frac{1+p}{2}}^{\theta} = C''_{n,p} \eta^{\frac{1}{2}(1-\theta)} \left(C'_{n,\frac{1+p}{2}}\right)^{\theta} = C_{n,p} \eta^{\frac{p-1}{p(3-p)}},$$

with $C_{n,p} = C''_{n,p} \left(C'_{n,\frac{1+p}{2}}\right)^{\theta}$, since $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{\frac{1+p}{2}}$ implies $1-\theta = \frac{2p-2}{p(3-p)}$. □

2.3.1. *Injectivity.* We can now prove Proposition 16.

Proof of Proposition 16. We have

$$S_\eta^{\mathcal{D}} f = \sum_{Q\in\mathcal{D}} \triangle_{Q;\kappa} S_\eta f = \sum_{Q\in\mathcal{D}} \langle S_\eta f, h_{Q;\kappa}\rangle h_{Q;\kappa} = \sum_{Q\in\mathcal{D}} \left\langle \sum_{I\in\mathcal{D}} \langle f, h_{I;\kappa}\rangle h_{I;\kappa}^\eta, h_{Q;\kappa} \right\rangle h_{Q;\kappa} = \sum_{Q,I\in\mathcal{D}} \langle f, h_{I;\kappa}\rangle \left\langle h_{I;\kappa}^\eta, h_{Q;\kappa} \right\rangle h_{Q;\kappa} ,$$

and

$$\left\|S_\eta^{\mathcal{D}}f\right\|_{L^p}\approx\left\|\left(\sum_{Q\in\mathcal{D}}\left|\left\langle S_\eta f,h_{Q;\kappa}\right\rangle h_{Q;\kappa}\right|^2\right)^{\frac{1}{2}}\right\|_{L^p}=\left\|\left(\sum_{Q\in\mathcal{D}}\left|\sum_{I\in\mathcal{D}}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa}^\eta,h_{Q;\kappa}\right\rangle h_{Q;\kappa}\right|^2\right)^{\frac{1}{2}}\right\|_{L^p}$$

$$\approx\left\|\left(\sum_{Q\in\mathcal{D}}\left|\left\langle f,h_{Q;\kappa}\right\rangle\left\langle h_{Q;\kappa}^\eta,h_{Q;\kappa}\right\rangle\right|^2\left|h_{Q;\kappa}\right|^2\right)^{\frac{1}{2}}\right\|_{L^p}+O\left(\left\|\left(\sum_{Q\in\mathcal{D}}\left|\sum_{I\in\mathcal{D}:\ I\neq Q}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa}^\eta,h_{Q;\kappa}\right\rangle\right|^2\left|h_{Q;\kappa}\right|^2\right)^{\frac{1}{2}}\right\|_{L^p}\right)$$

$$\approx\left\|\left(\sum_{Q\in\mathcal{D}}\left|\left\langle f,h_{Q;\kappa}\right\rangle\right|^2\frac{1}{|Q|}\mathbf{1}_Q\right)^{\frac{1}{2}}\right\|_{L^p}^p+O\left(\left\|\left(\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}\left|\sum_{I\in\mathcal{D}:\ I\neq Q}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa}^\eta,h_{Q;\kappa}\right\rangle\right|^2\mathbf{1}_Q\right)^{\frac{1}{2}}\right\|_{L^p}\right),$$

where by the square function estimate (2.1),

$$C_p\left\|f\right\|_{L^p}^p\geq\left\|\left(\sum_{Q\in\mathcal{D}}\left|\left\langle f,h_{Q;\kappa}\right\rangle\right|^2\frac{1}{|Q|}\mathbf{1}_Q\right)^{\frac{1}{2}}\right\|_{L^p}^p=\left\|\left(\sum_{Q\in\mathcal{D}}\left|\triangle_{Q;\kappa}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^p}^p\geq c_p\left\|f\right\|_{L^p}^p\ ,$$

for some $C_p,c_p>0$.

Thus we have for each $Q\in\mathcal{D}$,

$$\sum_{I\in\mathcal{D}:\ I\neq Q}\left\langle f,h_I\right\rangle\left\langle h_I^\eta,h_Q\right\rangle=\sum_{\substack{I\in\mathcal{D}:\ \ell(I)<\ell(Q)\\ I\in\mathrm{Car}(Q)}}\left\langle f,h_I\right\rangle\left\langle h_I^\eta,h_Q\right\rangle+\sum_{\substack{I\in\mathcal{D}:\ \ell(I)>\ell(Q)\\ Q\cap\mathcal{H}_{\frac{\eta}{2}}(I)\neq\emptyset}}\left\langle f,h_I\right\rangle\left\langle h_I^\eta,h_Q\right\rangle$$
$$+\sum_{\substack{I\in\mathcal{D}:\ \ell(I)\geq\ell(Q)\geq\eta\ell(I)\\ Q\in\mathrm{Car}(I)}}\left\langle f,h_I\right\rangle\left\langle h_I^\eta,h_Q\right\rangle.$$

As a consequence of the estimates in Lemma 18, we have for each $Q\in\mathcal{D}$,

$$\begin{aligned}\left|\sum_{I\in\mathcal{D}:\ I\neq Q}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa}^\eta,h_Q\right\rangle\right| &\lesssim \eta\sum_{\substack{I\in\mathcal{D}:\ \ell(I)<\ell(Q)\\ I\in\mathrm{Car}(Q)}}\left|\left\langle f,h_{I;\kappa}\right\rangle\right|\left(\frac{\ell(I)}{\ell(Q)}\right)^{\frac{n}{2}}+\sum_{\substack{I\in\mathcal{D}:\ \ell(Q)\leq\eta\ell(I)\\ Q\cap\mathcal{H}_{\frac{\eta}{2}}(I)\neq\emptyset}}\left|\left\langle f,h_{I;\kappa}\right\rangle\right|\frac{1}{\eta^\kappa}\left(\frac{\ell(Q)}{\ell(I)}\right)^{\kappa+\frac{n}{2}}\\ &\quad+\left|\sum_{\substack{I\in\mathcal{D}:\ \ell(I)\geq\ell(Q)\geq\eta\ell(I)\\ Q\in\mathrm{Car}(I)}}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa}^\eta,h_Q\right\rangle\right|\\ &\equiv A(Q)+B(Q)+C(Q).\end{aligned}$$

Altogether we have

$$\left\|\left(\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}\left|\sum_{I\in\mathcal{D}:\ I\neq Q}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa}^\eta,h_{Q;\kappa}\right\rangle\right|^2\mathbf{1}_Q\right)^{\frac{1}{2}}\right\|_{L^p}\lesssim\left\|\left(\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}A(Q)^2\mathbf{1}_Q\right)^{\frac{1}{2}}\right\|_{L^p}\tag{2.8}$$
$$+\left\|\left(\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}B(Q)^2\mathbf{1}_Q\right)^{\frac{1}{2}}\right\|_{L^p}+\left\|\left(\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}C(Q)^2\mathbf{1}_Q\right)^{\frac{1}{2}}\right\|_{L^p}.$$

We now claim that

$$\left\|\left(\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}\left|\sum_{I\in\mathcal{D}:\ I\neq Q}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa}^\eta,h_{Q;\kappa}\right\rangle\right|^2\mathbf{1}_Q\right)^{\frac{1}{2}}\right\|_{L^p}\lesssim\eta^{\frac{1}{2}\gamma_p}\left(\log_2\frac{1}{\eta}\right)\left\|f\right\|_{L^p}.\tag{2.9}$$

With this established, and since $\kappa > \frac{n}{2}$, we obtain

$$\left\| \left(\sum_{Q\in\mathcal{D}} \frac{1}{|Q|} \left| \sum_{I\in\mathcal{D}:\ I\neq Q} \langle f, h_{I;\kappa}\rangle \left\langle h^{\eta}_{I;\kappa}, h_{Q;\kappa}\right\rangle \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \leq C\eta^{\frac{1}{2}\gamma_p} \left(\log_2 \frac{1}{\eta}\right) \|f\|_{L^p} < \frac{c_p}{2} \|f\|_{L^p} \ ,$$

with $\eta > 0$ sufficiently small. This then gives

$$C_p \|f\|_{L^p} \geq \left\| S^{\mathcal{D}}_{\eta} f \right\|_{L^p} \geq c_p \|f\|_{L^p} - \frac{c_p}{2} \|f\|_{L^p} = \frac{c_p}{2} \|f\|_{L^p} \ ,$$

which completes the proof of Proposition 16 modulo (2.9).

We prove (2.9) by estimating each of the three terms on the right hand side of (2.8) separately, beginning with the term involving $A(Q)$.

Case $A(Q)$: For each $Q \in \mathcal{D}$, we have for $0 < \varepsilon < 1$ and $0 < \gamma < n - \varepsilon$,

$$\begin{aligned}
A(Q) &= \eta \sum_{\substack{I\in\mathcal{D}:\ \ell(I)<\ell(Q)\\ I\in\mathrm{Car}(Q)}} |\langle f, h_{I;\kappa}\rangle| \left(\frac{\ell(I)}{\ell(Q)}\right)^{\frac{n}{2}} = \eta \sum_{t=1}^{\infty} \sum_{\substack{I\in\mathcal{D}:\ \ell(I)=2^{-t}\ell(Q)\\ I\in\mathrm{Car}(Q)}} |\langle f, h_{I;\kappa}\rangle| \, 2^{-t\frac{n}{2}} \\
&\lesssim \eta \sum_{t=1}^{\infty} \sqrt{\sum_{\substack{I\in\mathcal{D}:\ \ell(I)=2^{-t}\ell(Q)\\ I\in\mathrm{Car}(Q)}} |\langle f, h_{I;\kappa}\rangle|^2 \, 2^{-t(n-\varepsilon)}} = \eta \sum_{t=1}^{\infty} 2^{-t\frac{n-\varepsilon-\gamma}{2}} \sqrt{\sum_{\substack{I\in\mathcal{D}:\ \ell(I)=2^{-t}\ell(Q)\\ I\in\mathrm{Car}(Q)}} 2^{-t\gamma} |\langle f, h_{I;\kappa}\rangle|^2} \\
&\leq \eta \sqrt{\sum_{t=1}^{\infty} 2^{-t(n-\varepsilon-\gamma)}} \sqrt{\sum_{t=1}^{\infty} \sum_{\substack{I\in\mathcal{D}:\ \ell(I)=2^{-t}\ell(Q)\\ I\in\mathrm{Car}(Q)}} 2^{-t\gamma} |\langle f, h_{I;\kappa}\rangle|^2} = \eta \sqrt{\frac{2^{-(n-\varepsilon-\gamma)}}{1-2^{-(n-\varepsilon-\gamma)}}} \sqrt{\sum_{t=1}^{\infty} \sum_{\substack{I\in\mathcal{D}:\ \ell(I)=2^{-t}\ell(Q)\\ I\in\mathrm{Car}(Q)}} 2^{-t\gamma} |\langle f, h_{I;\kappa}\rangle|^2}.
\end{aligned}$$

and so

$$A(Q) = \eta \sum_{\substack{I\in\mathcal{D}:\ \ell(I)<\ell(Q)\\ I\in\mathrm{Car}(Q)}} |\langle f, h_{I;\kappa}\rangle| \left(\frac{\ell(I)}{\ell(Q)}\right)^{\frac{n}{2}} \leq \eta \sqrt{\sum_{t=1}^{\infty} \sum_{\substack{I\in\mathcal{D}:\ \ell(I)=2^{-t}\ell(Q)\\ I\in\mathrm{Car}(Q)}} 2^{-t(n-2\varepsilon)} |\langle f, h_{I;\kappa}\rangle|^2}$$

if we take $\gamma = n - 2\varepsilon$. It follows that

$$\begin{aligned}
&\left\| \left(\sum_{Q\in\mathcal{D}} \frac{1}{|Q|} A(Q)^2 \mathbf{1}_Q \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \eta \left\| \left(\sum_{Q\in\mathcal{D}} \frac{1}{|Q|} \sum_{t=1}^{\infty} \sum_{\substack{I\in\mathcal{D}:\ \ell(I)=2^{-t}\ell(Q)\\ I\in\mathrm{Car}(Q)}} 2^{-t(n-2\varepsilon)} |\langle f, h_{I;\kappa}\rangle|^2 \mathbf{1}_Q \right) \right\|_{L^p} \\
&= \eta \left\| \left(\sum_{I\in\mathcal{D}} |\langle f, h_{I;\kappa}\rangle|^2 \sum_{t=1}^{\infty} \frac{1}{|Q|} \sum_{\substack{Q\in\mathcal{D}:\ \ell(I)=2^{-t}\ell(Q)\\ I\in\mathrm{Car}(Q)}} 2^{-t(n-2\varepsilon)} \mathbf{1}_Q \right)^{\frac{1}{2}} \right\|_{L^p} \\
&\leq \eta \left\| \left(\sum_{I\in\mathcal{D}} |\langle f, h_{I;\kappa}\rangle|^2 \sum_{t=1}^{\infty} \frac{1}{|2^t I|} 2^{-t(n-2\varepsilon)} \mathbf{1}_{2^t I} \right)^{\frac{1}{2}} \right\|_{L^p} \leq \eta \left\| \left(\sum_{I\in\mathcal{D}} \frac{|\langle f, h_{I;\kappa}\rangle|^2}{|I|} \sum_{t=1}^{\infty} 2^{-2tn+2\varepsilon t} \mathbf{1}_{2^t I} \right)^{\frac{1}{2}} \right\|_{L^p} \\
&\lesssim \eta \left\| \left(\sum_{I\in\mathcal{D}} \frac{|\langle f, h_{I;\kappa}\rangle|^2}{|I|} (M\mathbf{1}_I)^{2\frac{2-2\varepsilon}{2}} \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \eta \left\| \left(\sum_{I\in\mathcal{D}} \frac{|\langle f, h_{I;\kappa}\rangle|^2}{|I|} (M_r\mathbf{1}_I)^2 \right)^{\frac{1}{2}} \right\|_{L^p} \\
&\lesssim \eta \left\| \left(\sum_{I\in\mathcal{D}} \frac{|\langle f, h_{I;\kappa}\rangle|^2}{|I|} \mathbf{1}_I \right)^{\frac{1}{2}} \right\|_{L^p} \approx \eta \|f\|_{L^p} \,,
\end{aligned}$$

provided $1<r=\frac{2}{2-2\varepsilon}=\frac{1}{1-\varepsilon}<p$. Indeed,

$$\sum_{t=1}^{\infty}2^{-2tn+2\varepsilon t}\mathbf{1}_{2^tI}\lesssim\left(M\mathbf{1}_I\right)^{2\frac{2-2\varepsilon}{2}}=\left(M_r\mathbf{1}_I\right)^2,$$

where the inequality follows from

$$\begin{aligned}&\sum_{t=1}^{\infty}2^{-2tn+2\varepsilon t}\mathbf{1}_{2^tI}\left(x\right)\approx\sum_{t=1}^{\infty}2^{-2tn+2\varepsilon t}\mathbf{1}_{2^tI-2^{t-1}I}\left(x\right)\\ =&\sum_{t=1}^{\infty}2^{-2tn\left(1-\frac{\varepsilon}{n}\right)}\mathbf{1}_{2^tI-2^{t-1}I}\left(x\right)\lesssim\sum_{t=1}^{\infty}M\mathbf{1}_I\left(x\right)^{2\left(1-\frac{\varepsilon}{n}\right)}\mathbf{1}_{2^tI-2^{t-1}I}\left(x\right)=M\mathbf{1}_I\left(x\right)^{2\left(1-\frac{\varepsilon}{n}\right)},\end{aligned}$$

and the equality follows by definition of M_r and since $\mathbf{1}_I=\left(\mathbf{1}_I\right)^r$, namely

$$\left(M\mathbf{1}_I\right)^{2\frac{2-2\varepsilon}{2}}=\left(\left(M\left(\mathbf{1}_I\right)^r\right)^{\frac{1}{r}}\right)^2=\left(M_r\mathbf{1}_I\right)^2.$$

Case $B\left(Q\right)$: Set $\eta=2^{-\beta}$. Note that the function squared in the second norm in (2.8) then satisfies

$$\begin{aligned}&\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}B\left(Q\right)^2\mathbf{1}_Q\left(x\right)=\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}\left(\sum_{\substack{I\in\mathcal{D}:\ \ell(Q)\leq\eta\ell(I)\\ Q\cap\mathcal{H}_{\frac{\eta}{2}}(I)\neq\emptyset}}\left|\left\langle f,h_{I;\kappa}\right\rangle\right|\frac{1}{\eta^{\kappa}}\left(\frac{\ell\left(Q\right)}{\ell\left(I\right)}\right)^{\kappa+\frac{n}{2}}\right)^2\mathbf{1}_Q\left(x\right)\\ =&\ \frac{1}{\eta^{2\kappa}}\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}\sum_{\substack{I\in\mathcal{D}:\ \ell(Q)\leq\eta\ell(I)\\ Q\cap\mathcal{H}_{\frac{\eta}{2}}(I)\neq\emptyset}}\ \sum_{\substack{I'\in\mathcal{D}:\ \ell(Q)\leq\eta\ell(I')\\ Q\cap\mathcal{H}_{\frac{\eta}{2}}(I')\neq\emptyset}}\left|\left\langle f,h_{I;\kappa}\right\rangle\right|\left|\left\langle f,h_{I';\kappa}\right\rangle\right|\left(\frac{\ell\left(Q\right)}{\ell\left(I\right)}\right)^{\kappa+\frac{n}{2}}\left(\frac{\ell\left(Q\right)}{\ell\left(I'\right)}\right)^{\kappa+\frac{n}{2}}\mathbf{1}_Q\left(x\right)\\ =&\ \frac{1}{\eta^{2\kappa}}2\sum_{I,I'\in\mathcal{D}\text{ and }I\subset I'}\left|\left\langle f,h_{I;\kappa}\right\rangle\right|\left|\left\langle f,h_{I';\kappa}\right\rangle\right|\left(\frac{1}{\ell\left(I\right)\ell\left(I'\right)}\right)^{\kappa+\frac{n}{2}}\sum_{\substack{Q\in\mathcal{D}:\ \ell(Q)\leq\eta\ell(I)\\ Q\cap\mathcal{H}_{\frac{\eta}{2}}(I)\neq\emptyset}}\ell\left(Q\right)^{2\kappa}\mathbf{1}_Q\left(x\right)\\ \approx&\ \frac{1}{\eta^{2\kappa}}\sum_{I,I'\in\mathcal{D}\text{ and }I\subset I'}\left|\left\langle f,h_{I;\kappa}\right\rangle\right|\left|\left\langle f,h_{I';\kappa}\right\rangle\right|\left(\frac{1}{\ell\left(I\right)\ell\left(I'\right)}\right)^{\kappa+\frac{n}{2}}\ell\left(I\right)^{2\kappa}\sum_{t=\beta}^{\infty}\sum_{\substack{Q\in\mathcal{D}:\ \ell(Q)=2^{-t}\ell(I)\\ Q\cap\mathcal{H}_{\frac{\eta}{2}}(I)\neq\emptyset}}\mathbf{1}_Q\left(x\right)2^{-t2\kappa},\end{aligned}$$

where for $t\geq\beta$ and $x\in\mathcal{H}_{\frac{\eta}{2}}\left(I\right)$, we have

$$\sum_{\substack{Q\in\mathcal{D}:\ \ell(Q)=2^{-t}\ell(I)\\ Q\cap\mathcal{H}_{\frac{\eta}{2}}(I)\neq\emptyset}}\mathbf{1}_Q\left(x\right)\leq1,$$

so that

$$\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}B\left(Q\right)^2\mathbf{1}_Q\left(x\right)\lesssim\frac{1}{\eta^{2\kappa}}\sum_{I,I'\in\mathcal{D}\text{ and }I\subset I'}\left|\left\langle f,h_{I;\kappa}\right\rangle\right|\left|\left\langle f,h_{I';\kappa}\right\rangle\right|\left(\frac{1}{\ell\left(I\right)\ell\left(I'\right)}\right)^{\kappa+\frac{n}{2}}\ell\left(I\right)^{2\kappa}\sum_{t=\beta}^{\infty}2^{-t2\kappa}\mathbf{1}_{\mathcal{H}_{\frac{\eta}{2}}(I)}\left(x\right).$$

Now recalling $2^{-t}=\frac{\ell(Q)}{\ell(I)}$, we have for $t\geq\beta$,

$$\#\left\{Q\in\mathcal{D}:\ \operatorname{dist}\left(Q,\partial I\right)\geq\ell\left(Q\right)=2^{-t}\ell\left(I\right)\text{ and }Q\cap\mathcal{H}_{\frac{\eta}{2}}\left(I\right)\neq\emptyset\right\}\text{ is }\left\{\begin{array}{ccc}\approx\eta2^{tn}&\text{if}&t\geq\beta\\ 0&\text{if}&1\leq t<\beta\end{array}\right..$$

Our blanket assumption that $\kappa > \frac{n}{2}$ shows that all of the geometric series appearing below are convergent. Then we have

$$\begin{aligned}\sum_{Q\in\mathcal{D}_{\mathrm{good}}}\frac{1}{|Q|}B(Q)^2\mathbf{1}_Q(x) &\lesssim \frac{1}{\eta^{2\kappa}}\sum_{I,I'\in\mathcal{D}\text{ and }I\subset I'}\frac{|\langle f,h_{I;\kappa}\rangle|\,|\langle f,h_{I';\kappa}\rangle|}{\ell(I)^{\frac{n}{2}}\,\ell(I')^{\frac{n}{2}}}\left(\frac{\ell(I)}{\ell(I')}\right)^{\kappa}\sum_{t=\beta}^{\infty}2^{-t2\kappa}\mathbf{1}_{\mathcal{H}_{\frac{\eta}{2}}(I)}(x)\\ &\lesssim \frac{1}{\eta^{2\kappa}}\sum_{I,I'\in\mathcal{D}\text{ and }I\subset I'}\frac{|\langle f,h_{I;\kappa}\rangle|\,|\langle f,h_{I';\kappa}\rangle|}{\ell(I)^{\frac{n}{2}}\,\ell(I')^{\frac{n}{2}}}\left(\frac{\ell(I)}{\ell(I')}\right)^{\kappa}\frac{2^{-\beta2\kappa}}{1-2^{-2\kappa}}\mathbf{1}_{\mathcal{H}_{\frac{\eta}{2}}(I)}(x)\\ &\lesssim \sum_{I,I'\in\mathcal{D}\text{ and }I\subset I'}\frac{|\langle f,h_{I;\kappa}\rangle|\,|\langle f,h_{I';\kappa}\rangle|}{\ell(I)^{\frac{n}{2}}\,\ell(I')^{\frac{n}{2}}}\left(\frac{\ell(I)}{\ell(I')}\right)^{\kappa}\mathbf{1}_{\mathcal{H}_{\frac{\eta}{2}}(I)}(x),\end{aligned}$$

which in turn equals,

$$\begin{aligned}&\sum_{I\in\mathcal{D}}\sum_{s=1}^{\infty}\frac{|\langle f,h_{I;\kappa}\rangle|}{\sqrt{|I|}\ell}\frac{\left|\left\langle f,h_{(\pi^{(s)}I);\kappa}\right\rangle\right|}{\sqrt{\left|\pi^{(s)}I\right|}}\left(\frac{\ell(I)}{\ell\left(\pi^{(s)}I\right)}\right)^{\kappa}\mathbf{1}_{\mathcal{H}_{\eta}(I)}(x)\\ =\;&\sum_{I\in\mathcal{D}}\sum_{s=1}^{\infty}\frac{|\langle f,h_{I;\kappa}\rangle|}{|I|^{\frac{1}{2}}}\frac{\left|\left\langle f,h_{(\pi^{(s)}I);\kappa}\right\rangle\right|}{\left|\pi^{(s)}I\right|^{\frac{1}{2}}}2^{-s\kappa}\mathbf{1}_{\mathcal{H}_{\eta}(I)}(x)\\ =\;&\left(\sum_{s=1}^{\infty}2^{-s\kappa}\right)\sum_{I\in\mathcal{D}}\frac{|\langle f,h_{I;\kappa}\rangle|}{|I|^{\frac{1}{2}}}\frac{\left|\left\langle f,h_{(\pi^{(s)}I);\kappa}\right\rangle\right|}{\left|\pi^{(s)}I\right|^{\frac{1}{2}}}\mathbf{1}_{\mathcal{H}_{\eta}(I)}(x),\end{aligned}$$

which is at most

$$\left(\sum_{s=1}^{\infty}2^{-s\kappa}\right)\sqrt{\sum_{I\in\mathcal{D}}\left(\frac{|\langle f,h_{I;\kappa}\rangle|}{|I|^{\frac{1}{2}}}\right)^2\mathbf{1}_{\mathcal{H}_{\eta}(I)}(x)}\sqrt{\sum_{I\in\mathcal{D}}\left(\frac{\left|\left\langle f,h_{(\pi^{(s)}I);\kappa}\right\rangle\right|}{\left|\pi^{(s)}I\right|^{\frac{1}{2}}}\right)^2\mathbf{1}_{\mathcal{H}_{\eta}\left(\pi^{(t)}I\right)}(x)}\approx\sum_{I\in\mathcal{D}}\left(\frac{|\langle f,h_{I;\kappa}\rangle|}{|I|^{\frac{1}{2}}}\right)^2\mathbf{1}_{\mathcal{H}_{\eta}(I)}(x).$$

By Lemma 19 we thus have

$$\left\|\left(\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}B(Q)^2\mathbf{1}_Q\right)^{\frac{1}{2}}\right\|_{L^p}\lesssim\left\|\left(\sum_{I\in\mathcal{D}}\left(\frac{|\langle f,h_{I;\kappa}\rangle|}{|I|^{\frac{1}{2}}}\mathbf{1}_{\mathcal{H}_{\eta}(I)}(x)\right)^2\right)^{\frac{1}{2}}\right\|_{L^p}\le C_{p,n}\eta^{\frac{1}{2(p-1)}}\|f\|_{L^p}\;. \tag{2.10}$$

Case $C(Q)$: We have,

$$\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}C(Q)^2\mathbf{1}_Q=\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}\left|\sum_{\substack{I\in\mathcal{D}:\ \ell(I)\geq\ell(Q)\geq\eta\ell(I)\\ Q\in\mathrm{Car}(I)}}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}\right\rangle\right|^2\mathbf{1}_Q(x)$$

$$=\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}\left(\sum_{\substack{I\in\mathcal{D}:\ \ell(I)\geq\ell(Q)\geq\eta\ell(I)\\ Q\in\mathrm{Car}(I)}}\ \sum_{\substack{I\in\mathcal{D}:\ \ell(I')\geq\ell(Q)\geq\eta\ell(I')\\ Q\in\mathrm{Car}(I')}}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}\right\rangle\left\langle f,h_{I';\kappa}\right\rangle\left\langle h_{I';\kappa}^{\eta},h_{Q}\right\rangle\right)\mathbf{1}_Q(x)$$

$$\approx\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}\left(\sum_{\substack{I,I'\in\mathcal{D}:\ I\subset I'\text{ and }\ell(I)\geq\ell(Q)\geq\eta\ell(I')\\ Q\in\mathrm{Car}(I)\cap\mathrm{Car}(I')}}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}\right\rangle\left\langle f,h_{I';\kappa}\right\rangle\left\langle h_{I';\kappa}^{\eta},h_{Q;\kappa}\right\rangle\right)\mathbf{1}_Q(x)$$

$$=\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}\left(\sum_{\substack{I,I'\in\mathcal{D}:\ I\subset I'\\ Q\in\mathrm{Car}(I)\cap\mathrm{Car}(I')\\ \ell(I)\geq\ell(Q)\geq\eta\ell(I')}}\left\langle f,h_I\right\rangle\left\langle f,h_{I'}\right\rangle\left\langle h_I^{\eta},h_Q\right\rangle\left\langle h_{I'}^{\eta},h_Q\right\rangle\right)\mathbf{1}_Q(x).$$

We first compute the diagonal sum restricted to $I=I'$. Set

$$\Gamma_{\eta,t}(I)\equiv\left\{x\in I:\mathrm{dist}\left(x,\mathcal{H}_\eta(I)\right)\approx 2^t\eta\ell(I)\right\},\qquad\text{for }0\leq t\leq\beta,$$

where we recall that $\eta=2^{-\beta}$, and note that the diagonal portion of the sum above equals

$$\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}\left(\sum_{\substack{I\in\mathcal{D}:\ Q\in\mathrm{Car}(I)\\ \ell(I)\geq\ell(Q)\geq\eta\ell(I)}}\left|\left\langle f,h_I\right\rangle\right|^2\left|\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}\right\rangle\right|^2\right)\mathbf{1}_Q(x)=\sum_{I\in\mathcal{D}}\left|\left\langle f,h_{I;\kappa}\right\rangle\right|^2\sum_{\substack{Q\in\mathcal{D}:\ Q\in\mathrm{Car}(I)\\ \ell(I)\geq\ell(Q)\geq\eta\ell(I)}}\frac{\left|\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}\right\rangle\right|^2}{|Q|}\mathbf{1}_Q(x)$$

$$\lesssim\sum_{I\in\mathcal{D}}\left|\left\langle f,h_{I;\kappa}\right\rangle\right|^2\sum_{\substack{Q\in\mathcal{D}:\ Q\in\mathrm{Car}(I)\\ \ell(I)\geq\ell(Q)\geq\eta\ell(I)}}\frac{\eta^2\left(\frac{\ell(Q)}{\ell(I)}\right)^{n-2}}{\ell(Q)^n}\mathbf{1}_Q(x)=\eta^2\sum_{I\in\mathcal{D}}\left|\left\langle f,h_{I;\kappa}\right\rangle\right|^2\sum_{\substack{Q\in\mathcal{D}:\ Q\in\mathrm{Car}(I)\\ \ell(I)\geq\ell(Q)\geq\eta\ell(I)}}\frac{1}{\ell(I)^{n-2}\ell(Q)^2}\mathbf{1}_Q(x)$$

$$\approx\eta^2\sum_{I\in\mathcal{D}}\left|\left\langle f,h_{I;\kappa}\right\rangle\right|^2\frac{1}{\ell(I)^{n-2}\left[\eta\ell(I)+\mathrm{dist}\left(x,\mathcal{H}_\eta(I)\right)\right]^2}\mathbf{1}_I(x)=\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|^2}{|I|}\left(\frac{1}{1+\frac{\mathrm{dist}(x,\mathcal{H}_\eta(I))}{\eta\ell(I)}}\right)^2\mathbf{1}_I(x),$$

which can be written as

$$\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|^2}{|I|}\sum_{t=0}^{\beta}\mathbf{1}_{\Gamma_{\eta,t}}(x)\left(\frac{1}{1+\frac{\mathrm{dist}(x,\mathcal{H}_\eta(I))}{\eta\ell(I)}}\right)^2\approx\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|^2}{|I|}\sum_{t=0}^{\beta}2^{-2t}\mathbf{1}_{\Gamma_{\eta,t}(I)}(x).$$

Thus

$$\left\|\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}\left(\sum_{\substack{I\in\mathcal{D}:\ Q\in\mathrm{Car}(I)\\ \ell(I)\geq\ell(Q)\geq\eta\ell(I)}}\left|\left\langle f,h_{I;\kappa}\right\rangle\right|^2\left|\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}\right\rangle\right|^2\right)\mathbf{1}_Q(x)\right\|_{L^p}\lesssim\sum_{t=0}^{\beta}2^{-2t}\left\|\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|^2}{|I|}\mathbf{1}_{\Gamma_{\eta,t}(I)}(x)\right\|_{L^p}.$$

From the estimate for term B in (2.10), with η replaced by $2^t\eta$, we obtain

$$\left\|\sum_{I\in\mathcal{D}}\frac{\left|\langle f,h_{I;\kappa}\rangle\right|^2}{|I|}\mathbf{1}_{\Gamma_{\eta,t}(I)}(x)\right\|_{L^p}\lesssim C_{p,n}\left(2^t\eta\right)^{\frac{1}{2(p-1)}}\|f\|_{L^p}\ ,$$

and so altogether, the diagonal portion of $\left\|\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}C(Q)^2\mathbf{1}_Q(x)\right\|_{L^p}$ is at most

$$\begin{aligned}
&\sum_{t=0}^{\beta}2^{-2t}\left\|\sum_{I\in\mathcal{D}}\frac{\left|\langle f,h_{I;\kappa}\rangle\right|^2}{|I|}\mathbf{1}_{\Gamma_{\eta,t}(I)}(x)\right\|_{L^p}\lesssim\sum_{t=0}^{\beta}C_{p,n}2^{-2t}\left(2^t\eta\right)^{\frac{1}{2(p-1)}}\|f\|_{L^p}\\
=\ &\eta^{\frac{1}{2(p-1)}}\sum_{t=0}^{\beta}C_{p,n}2^{-t\left(2-\frac{1}{2(p-1)}\right)}\|f\|_{L^p}=\eta^{\frac{1}{2(p-1)}}\sum_{t=0}^{\beta}C_{p,n}2^{-t\frac{4(p-1)-1}{2(p-1)}}\|f\|_{L^p}\\
=\ &\eta^{\frac{1}{2(p-1)}}\sum_{t=0}^{\beta}C_{p,n}2^{-t\frac{4p-5}{2p-2}}\|f\|_{L^p}\approx C_{p,n}\left\{\begin{array}{ccc}\eta^{\frac{1}{2(p-1)}}\|f\|_{L^p} & \text{if} & p>\frac{5}{4}\\ \eta^2\left(\log_2\frac{1}{\eta}\right)\|f\|_{L^p} & \text{if} & p=\frac{5}{4}\\ \eta^2\|f\|_{L^p} & \text{if} & 1<p<\frac{5}{4}\end{array}\right. .
\end{aligned}$$

Now we use the estimate $\left|\left\langle h^\eta_{I;\kappa},h_{Q;\kappa}\right\rangle\right|\lesssim\eta\left(\frac{\ell(Q)}{\ell(I)}\right)^{\frac{n}{2}-1}$ for $Q\in\operatorname{Car}(I)$ and $\ell(Q)\geq\eta\ell(I)$, see the third line of Lemma 18, to obtain

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}}\frac{1}{|Q|}\left|\sum_{\substack{I\in\mathcal{D}:\ \ell(I)\geq\ell(Q)\geq\eta\ell(I)\\ Q\in\operatorname{Car}(I)}}\langle f,h_{I;\kappa}\rangle\langle h^\eta_I,h_Q\rangle\right|^2\mathbf{1}_Q(x)\\
\lesssim\ &\sum_{I,I'\in\mathcal{D}:\ I\subset I'}\left|\langle f,h_{I;\kappa}\rangle\right|\left|\langle f,h_{I';\kappa}\rangle\right|\sum_{\substack{Q\in\mathcal{D}:\ Q\in\operatorname{Car}(I)\cap\operatorname{Car}(I')\\ \ell(I)\geq\ell(Q)\geq\eta\ell(I')}}\left|\left\langle h^\eta_{I;\kappa},h_{Q;\kappa}\right\rangle\right|\left|\left\langle h^\eta_{I';\kappa},h_{Q;\kappa}\right\rangle\right|\frac{1}{|Q|}\mathbf{1}_Q(x)\\
\lesssim\ &\eta^2\sum_{I,I'\in\mathcal{D}:\ I\subset I'}\left|\langle f,h_{I;\kappa}\rangle\right|\left|\langle f,h_{I';\kappa}\rangle\right|\sum_{\substack{Q\in\mathcal{D}:\ Q\in\operatorname{Car}(I)\cap\operatorname{Car}(I')\\ \ell(I)\geq\ell(Q)\geq\eta\ell(I')}}\left(\frac{\ell(Q)}{\ell(I)}\right)^{\frac{n}{2}-1}\left(\frac{\ell(Q)}{\ell(I')}\right)^{\frac{n}{2}-1}\frac{1}{|Q|}\mathbf{1}_Q(x)\\
=\ &\eta^2\sum_{I,I'\in\mathcal{D}:\ I\subset I'}\frac{\left|\langle f,h_{I;\kappa}\rangle\right|\left|\langle f,h_{I';\kappa}\rangle\right|}{\sqrt{|I|}\sqrt{|I'|}}\sum_{\substack{Q\in\operatorname{Car}(I)\cap\operatorname{Car}(I')\\ \ell(I)\geq\ell(Q)\geq\eta\ell(I')}}\frac{\ell(I)}{\ell(Q)}\frac{\ell(I')}{\ell(Q)}\mathbf{1}_Q(x).
\end{aligned}$$

At this point we observe that the conditions imposed on the cubes I and I' in the sum above are that there exists a cube Q such that $Q\subset I\subset I'$, $Q\in\operatorname{Car}(I)\cap\operatorname{Car}(I')$, and $\ell(I)\geq\ell(Q)\geq\eta\ell(I')$. It follows from these conditions that

$$I\in\operatorname{Car}(I')\ \text{ and }\ell(I)\leq\ell(I')\leq\frac{1}{\eta}\ell(I)=2^\beta\ell(I).$$

Thus we can now pigeonhole the ratio of the lengths of I and I' by

$$\frac{\ell(I')}{\ell(I)}=2^s,\qquad\text{for }0\leq s\leq\beta.$$

With s fixed we have $I'=\pi^{(s)}I$ and

$$\begin{aligned}
&\eta^{2}\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|\left|\left\langle f,h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|}{\sqrt{\left|I\right|}\sqrt{\left|\pi^{(s)}I\right|}}\sum_{\substack{Q\in\operatorname{Car}(I)\cap\operatorname{Car}\left(\pi^{(s)}I\right)\\ \ell(I)\geq\ell(Q)\geq\eta\ell\left(\pi^{(s)}I\right)}}\frac{\ell\left(I\right)}{\ell\left(Q\right)}\frac{\ell\left(\pi^{(s)}I\right)}{\ell\left(Q\right)}\mathbf{1}_{Q}\left(x\right)\\
=\ &\eta^{2}\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|\left|\left\langle f,h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|}{\sqrt{\left|I\right|}\sqrt{\left|\pi^{(s)}I\right|}}\sum_{\substack{Q\in\operatorname{Car}(I)\cap\operatorname{Car}\left(\pi^{(s)}I\right)\\ \ell(I)\geq\ell(Q)\geq2^{s}\eta\ell(I)}}2^{s}\left(\frac{\ell\left(I\right)}{\ell\left(Q\right)}\right)^{2}\mathbf{1}_{Q}\left(x\right)\\
\approx\ &2^{s}\eta^{2}\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|\left|\left\langle f,h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|}{\sqrt{\left|I\right|}\sqrt{\left|\pi^{(s)}I\right|}}2^{s}\left(\frac{\ell\left(I\right)}{2^{s}\eta\ell\left(I\right)+\operatorname{dist}\left(x,\mathcal{H}_{2^{s}\eta}\left(I\right)\right)}\right)^{2}\mathbf{1}_{I}\left(x\right)\\
=\ &2^{s}\eta^{2}\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|\left|\left\langle f,h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|}{\sqrt{\left|I\right|}\sqrt{\left|\pi^{(s)}I\right|}}2^{s}\left(\frac{1}{2^{s}\eta+\frac{\operatorname{dist}\left(x,\mathcal{H}_{2^{s}\eta}(I)\right)}{\ell(I)}}\right)^{2}\mathbf{1}_{I}\left(x\right)\\
=\ &\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|}{\sqrt{\left|I\right|}}\frac{\left|\left\langle f,h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|}{\sqrt{\left|\pi^{(s)}I\right|}}\left(\frac{1}{1+\frac{\operatorname{dist}\left(x,\mathcal{H}_{2^{s}\eta}(I)\right)}{2^{s}\eta\ell(I)}}\right)^{2}\mathbf{1}_{I}\left(x\right),
\end{aligned}$$

where our sum is exactly like the diagonal portion with two exceptions, namely that I has been replaced by $\pi^{(s)}I$ in the second factor, and η has been replaced by $2^{s}\eta$ in the third factor. Thus we continue with,

$$\begin{aligned}
&\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|}{\sqrt{\left|I\right|}}\frac{\left|\left\langle f,h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|}{\sqrt{\left|\pi^{(s)}I\right|}}\left(\frac{1}{1+\frac{\operatorname{dist}\left(x,\mathcal{H}_{2^{s}\eta}(I)\right)}{2^{s}\eta\ell(I)}}\right)^{2}\mathbf{1}_{I}\left(x\right)\\
=\ &\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|}{\sqrt{\left|I\right|}}\frac{\left|\left\langle f,h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|}{\sqrt{\left|\pi^{(s)}I\right|}}\sum_{t=0}^{\beta-s}\mathbf{1}_{\Gamma_{2^{s}\eta,t}(I)}\left(x\right)\left(\frac{1}{1+\frac{\operatorname{dist}\left(x,\mathcal{H}_{2^{s}\eta}(I)\right)}{2^{s}\eta\ell(I)}}\right)^{2}\\
\approx\ &\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|}{\sqrt{\left|I\right|}}\frac{\left|\left\langle f,h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|}{\sqrt{\left|\pi^{(s)}I\right|}}\sum_{t=0}^{\beta-s}2^{-2t}\mathbf{1}_{\Gamma_{2^{s}\eta,t}(I)}\left(x\right),
\end{aligned}$$

since $\Gamma_{2^{s}\eta,t}\left(I\right)=\left\{x\in I:\operatorname{dist}\left(x,\mathcal{H}_{2^{s}\eta}\left(I\right)\right)\approx2^{t}2^{s}\eta\ell\left(I\right)\right\}$ and $\operatorname{dist}\left(x,\mathcal{H}_{2^{s}\eta}\left(I\right)\right)\leq\ell\left(I\right)$.

Now we continue to proceed as in the diagonal case to obtain,

$$\begin{aligned}
&\left\Vert\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|}{\sqrt{\left|I\right|}}\frac{\left|\left\langle f,h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|}{\sqrt{\left|\pi^{(s)}I\right|}}\left(\frac{1}{1+\frac{\operatorname{dist}\left(x,\mathcal{H}_{2^{s}\eta}(I)\right)}{2^{s}\eta\ell(I)}}\right)^{2}\mathbf{1}_{I}\right\Vert_{L^{p}}\\
\lesssim\ &\sum_{t=0}^{\beta-s}2^{-2t}\left\Vert\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|}{\sqrt{\left|I\right|}}\frac{\left|\left\langle f,h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|}{\sqrt{\left|\pi^{(s)}I\right|}}\mathbf{1}_{\Gamma_{2^{s}\eta,t}(I)}\right\Vert_{L^{p}}\\
\lesssim\ &\sum_{t=0}^{\beta-s}2^{-2t}\left\Vert\sqrt{\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|^{2}}{\left|I\right|}\mathbf{1}_{\Gamma_{2^{s}\eta,t}(I)}}\sqrt{\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|^{2}}{\left|\pi^{(s)}I\right|}\mathbf{1}_{\Gamma_{2^{s}\eta,t}(I)}}\right\Vert_{L^{p}}\\
\lesssim\ &\sum_{t=0}^{\beta-s}2^{-2t}\left\Vert\delta\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}\right\rangle\right|^{2}}{\left|I\right|}\mathbf{1}_{\Gamma_{2^{s}\eta,t}(I)}+\frac{1}{\delta}\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|^{2}}{\left|\pi^{(s)}I\right|}\mathbf{1}_{\Gamma_{2^{s}\eta,t}(I)}\right\Vert_{L^{p}},
\end{aligned}$$

for every choice of $\delta \in (0,1)$. Thus it remains to estimate each of the terms

$$\delta \sum_{t=0}^{\beta-s} 2^{-2t} \left\| \sum_{I\in\mathcal{D}} \frac{|\langle f, h_{I;\kappa}\rangle|^2}{|I|} \mathbf{1}_{\Gamma_{2^s\eta,t}(I)} \right\|_{L^p} \text{ and } \frac{1}{\delta} \sum_{t=0}^{\beta-s} 2^{-2t} \left\| \sum_{I\in\mathcal{D}} \frac{\left|\left\langle f, h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|^2}{\left|\pi^{(s)}I\right|} \mathbf{1}_{\Gamma_{2^s\eta,t}(I)} \right\|_{L^p},$$

and then minimize the sum over $0<\delta<1$. But from (2.10), we have

$$\begin{aligned} \sum_{t=0}^{\beta-s} 2^{-2t} \left\| \sum_{I\in\mathcal{D}} \frac{|\langle f, h_{I;\kappa}\rangle|^2}{|I|} \mathbf{1}_{\Gamma_{2^s\eta,t}(I)} \right\|_{L^p} &\lesssim C_{p,n} \left(2^s\eta\right)^{\frac{1}{2(p-1)}} \|f\|_{L^p} , \\ \sum_{t=0}^{\beta-s} 2^{-2t} \left\| \sum_{I\in\mathcal{D}} \frac{\left|\left\langle f, h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|^2}{\left|\pi^{(s)}I\right|} \mathbf{1}_{\Gamma_{2^s\eta,t}(I)} \right\|_{L^p} &\lesssim \sum_{t=0}^{\beta-s} 2^{-2t} \left\| \sum_{I'\in\mathcal{D}} \frac{|\langle f, h_{I';\kappa}\rangle|^2}{|I'|} \mathbf{1}_{\Gamma_{\eta,t}(I')} \right\|_{L^p} \lesssim C_{p,n} \eta^{\frac{1}{2(p-1)}} \|f\|_{L^p} , \end{aligned}$$

since

$$\begin{aligned} \Gamma_{\eta,t}\left(I'\right) \Gamma_{2^s\eta,t}\left(I\right) &= \left\{x\in I : \operatorname{dist}\left(x, \mathcal{H}_{2^s\eta}\left(I\right)\right) \approx 2^t 2^s \eta \ell\left(I\right)\right\} \\ &\subset \left\{x\in I' : \operatorname{dist}\left(x, \mathcal{H}_{\eta}\left(I'\right)\right) \approx 2^t \eta \ell\left(I'\right)\right\} = \Gamma_{\eta,t}\left(I'\right). \end{aligned}$$

Thus with $\delta = 2^{-\frac{s}{4(p-1)}}$, we obtain

$$\begin{aligned} & \left\| \sum_{I\in\mathcal{D}} \frac{|\langle f, h_{I;\kappa}\rangle|}{\sqrt{|I|}} \frac{\left|\left\langle f, h_{\left(\pi^{(s)}I\right);\kappa}\right\rangle\right|}{\sqrt{\left|\pi^{(s)}I\right|}} \left(\frac{1}{1+\frac{\operatorname{dist}\left(x,\mathcal{H}_{2^s\eta}(I)\right)}{2^s\eta\ell(I)}} \right)^2 \mathbf{1}_I \right\|_{L^p} \\ \lesssim\ & \delta C_{p,n} \left(2^s\eta\right)^{\frac{1}{2(p-1)}} \|f\|_{L^p} + \frac{1}{\delta} C_{p,n} \eta^{\frac{1}{2(p-1)}} \|f\|_{L^p} \\ =\ & \left[\delta 2^{\frac{s}{2(p-1)}} + \frac{1}{\delta} \right] C_{p,n} \eta^{\frac{1}{2(p-1)}} \|f\|_{L^p} = 2C_{p,n} 2^{\frac{s}{4(p-1)}} 2^{-\frac{\beta}{2(p-1)}} \|f\|_{L^p} \\ \leq\ & 2C_{p,n} 2^{-\frac{\beta}{4(p-1)}} \|f\|_{L^p} = 2C_{p,n} \eta^{\frac{1}{4(p-1)}} \|f\|_{L^p} , \end{aligned}$$

since $0\le s\le\beta$. Finally we sum in s from 0 to $\beta = \log_2 \frac{1}{\eta}$ to conclude that,

$$\left\| \left(\sum_{Q\in\mathcal{D}} \frac{1}{|Q|} C\left(Q\right)^2 \mathbf{1}_Q \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \eta^{\frac{1}{4(p-1)}} \log_2 \frac{1}{\eta} \|f\|_{L^p} .$$

This finishes the proof of (2.9) and hence the proof of Proposition 16. □

2.3.2. *Surjectivity.* The proof of Proposition 17 is very similar to that of the previous proposition in light of the following equivalences. Using $\left|\triangle_{I;\kappa}^{\eta} f\right| \le M^{\mathrm{dy}}\left(\triangle_{I;\kappa}^{\eta} f\right)$, together with the Fefferman-Stein vector-valued maximal inequalities [FeSt] and the square function equivalence (2.1), shows that

$$\left\| \left(\sum_{I\in\mathcal{D}} \left|\triangle_{I;\kappa}^{\eta} f\right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \approx \left\| \left(\sum_{I\in\mathcal{D}} \left|\triangle_{I;\kappa} f\right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \approx \left\| \sum_{I\in\mathcal{D}} \triangle_{I;\kappa} f \right\|_{L^p} = \|f\|_{L^p} .$$

We also have from the square function equivalence that

$$\left\| \left(\sum_{I\in\mathcal{D}} \left| \left(\triangle_{I;\kappa}^{\eta}\right)^{\mathrm{tr}} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} = \left\| \left(\sum_{I\in\mathcal{D}} \left| \left\langle f, h_{I;\kappa}^{\eta} \right\rangle h_{I;\kappa} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \approx \left\| \sum_{I\in\mathcal{D}} \left\langle f, h_{I;\kappa}^{\eta} \right\rangle h_{I;\kappa} \right\|_{L^p} = \left\| \sum_{I\in\mathcal{D}} \left(\triangle_{I;\kappa}^{\eta}\right)^{\mathrm{tr}} f \right\|_{L^p} . \tag{2.11}$$

Furthermore, from the definition $\left(S_{\eta}^{\mathcal{D}}\right)^{\operatorname{tr}} f=\sum_{I\in\mathcal{D}}\left\langle f,h_{I;\kappa}^{\eta}\right\rangle h_{I;\kappa}$, we then obtain

$$\left\Vert \left(S_{\eta}^{\mathcal{D}}\right)^{\operatorname{tr}} f\right\Vert_{L^{p}}\approx\left\Vert\left(\sum_{Q\in\mathcal{D}}\left\vert\triangle_{Q;\kappa}\left(S_{\eta}^{\mathcal{D}}\right)^{\operatorname{tr}} f\right\vert^{2}\right)^{\frac{1}{2}}\right\Vert_{L^{p}}=\left\Vert\left(\sum_{Q\in\mathcal{D}}\left\vert\left\langle\left(S_{\eta}^{\mathcal{D}}\right)^{\ast} f,h_{Q;\kappa}\right\rangle h_{Q;\kappa}\right\vert^{2}\right)^{\frac{1}{2}}\right\Vert_{L^{p}} \tag{2.12}$$
$$=\left\Vert\left(\sum_{Q\in\mathcal{D}}\frac{1}{\left\vert Q\right\vert}\left\vert\left\langle\sum_{I\in\mathcal{D}}\left\langle f,h_{I;\kappa}^{\eta}\right\rangle h_{I;\kappa},h_{Q;\kappa}\right\rangle\right\vert^{2}\right)^{\frac{1}{2}}\right\Vert_{L^{p}}=\left\Vert\left(\sum_{Q\in\mathcal{D}}\frac{1}{\left\vert Q\right\vert}\left\vert\left\langle f,h_{Q;\kappa}^{\eta}\right\rangle\right\vert^{2}\right)^{\frac{1}{2}}\right\Vert_{L^{p}}.$$

Proof of Proposition 17. From (2.12) we have,

$$\left\Vert\left(S_{\eta}^{\mathcal{D}}\right)^{\operatorname{tr}} f\right\Vert_{L^{p}}\approx\left\Vert\left(\sum_{Q\in\mathcal{D}}\frac{1}{\left\vert Q\right\vert}\left\vert\left\langle f,h_{Q;\kappa}^{\eta}\right\rangle\right\vert^{2}\right)^{\frac{1}{2}}\right\Vert_{L^{p}}=\left\Vert\left(\sum_{Q\in\mathcal{D}}\frac{1}{\left\vert Q\right\vert}\left\vert\sum_{I\in\mathcal{D}}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa},h_{Q;\kappa}^{\eta}\right\rangle\right\vert^{2}\right)^{\frac{1}{2}}\right\Vert_{L^{p}},$$

which we now compare to

$$\left\Vert S_{\eta}^{\mathcal{D}}f\right\Vert_{L^{p}}\approx\left\Vert\left(\sum_{Q\in\mathcal{D}}\left\vert\left\langle S_{\eta}f,h_{Q;\kappa}\right\rangle h_{Q;\kappa}\right\vert^{2}\right)^{\frac{1}{2}}\right\Vert_{L^{p}}=\left\Vert\left(\sum_{Q\in\mathcal{D}}\frac{1}{\left\vert Q\right\vert}\left\vert\sum_{I\in\mathcal{D}}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}\right\rangle\right\vert^{2}\right)^{\frac{1}{2}}\right\Vert_{L^{p}},$$

that was shown to be comparable to $\left\Vert f\right\Vert_{L^{p}}$ in Proposition 16 above. The only difference between the two right hand sides is that the convolution appears with $h_{Q;\kappa}^{\eta}$ in the first norm, and with $h_{I;\kappa}^{\eta}$ in the second norm. We now use the estimates in Lemma 18 just as in the proof of Proposition 16 above. Here is a sketch of the details that is virtually verbatim that of those in the proof of Proposition 16. Recall that $\mathcal{H}_{\eta}\left(I\right)$ is defined in (2.4).

For convenience we first rewrite the estimates in Lemma 18 so as to apply directly to the inner product $\left\langle h_{I;\kappa},h_{Q;\kappa}^{\eta}\right\rangle$ instead of $\left\langle h_{I;\kappa}^{\eta},h_{Q;\kappa}\right\rangle$. This is accomplished by simply interchanging Q and I throughout:

$$\begin{aligned}
\left\vert\left\langle h_{Q;\kappa}^{\eta},h_{Q;\kappa}\right\rangle\right\vert &\approx 1\text{ and }\left\vert\left\langle h_{Q;\kappa}^{\eta},h_{Q^{\prime};\kappa}\right\rangle\right\vert\lesssim\eta,\qquad\text{for }Q\text{ and }Q^{\prime}\text{ siblings},\\
\left\vert\left\langle h_{Q;\kappa}^{\eta},h_{I;\kappa}\right\rangle\right\vert &\lesssim\eta\left(\frac{\ell\left(Q\right)}{\ell\left(I\right)}\right)^{\frac{n}{2}},\qquad\text{for }Q\in\operatorname{Car}\left(I\right),\\
\left\vert\left\langle h_{Q;\kappa}^{\eta},h_{I;\kappa}\right\rangle\right\vert &\lesssim\eta\left(\frac{\ell\left(I\right)}{\ell\left(Q\right)}\right)^{\frac{n}{2}-1},\qquad\text{for }I\in\operatorname{Car}\left(Q\right)\text{ and }\ell\left(I\right)\geq\eta\ell\left(Q\right),\\
\left\vert\left\langle h_{Q;\kappa}^{\eta},h_{I;\kappa}\right\rangle\right\vert &\lesssim\frac{1}{\eta^{\kappa}}\left(\frac{\ell\left(I\right)}{\ell\left(Q\right)}\right)^{\kappa+\frac{n}{2}},\qquad\text{for }\ell\left(I\right)\leq\eta\ell\left(Q\right)\text{ and }I\cap\mathcal{H}_{\frac{\eta}{2}}\left(I\right)\neq\emptyset,\\
\left\langle h_{Q;\kappa}^{\eta},h_{I;\kappa}\right\rangle &= 0,\qquad\text{in all other cases.}
\end{aligned} \tag{2.13}$$

Now we have by the square function estimate (2.1),

$$\left\Vert\left(S_{\eta}^{\mathcal{D}}\right)^{\operatorname{tr}} f\right\Vert_{L^{p}}\approx\left\Vert\left(\sum_{Q\in\mathcal{D}}\left\vert\sum_{I\in\mathcal{D}}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa},h_{Q;\kappa}^{\eta}\right\rangle h_{Q;\kappa}\right\vert^{2}\right)^{\frac{1}{2}}\right\Vert_{L^{p}}$$
$$\approx\left\Vert\left(\sum_{Q\in\mathcal{D}}\left\vert\left\langle f,h_{Q;\kappa}\right\rangle\left\langle h_{Q;\kappa},h_{Q;\kappa}^{\eta}\right\rangle\right\vert^{2}\left\vert h_{Q;\kappa}\right\vert^{2}\right)^{\frac{1}{2}}\right\Vert_{L^{p}}+O\left(\left\Vert\left(\sum_{Q\in\mathcal{D}}\left\vert\sum_{I\in\mathcal{D}:\ I\neq Q}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa},h_{Q;\kappa}^{\eta}\right\rangle\right\vert^{2}\left\vert h_{Q}\right\vert^{2}\right)^{\frac{1}{2}}\right\Vert_{L^{p}}\right)$$
$$\approx\left\Vert\left(\sum_{Q\in\mathcal{D}}\left\vert\left\langle f,h_{Q;\kappa}\right\rangle\right\vert^{2}\frac{1}{\left\vert Q\right\vert}\mathbf{1}_{Q}\right)^{\frac{1}{2}}\right\Vert_{L^{p}}^{p}+O\left(\left\Vert\left(\sum_{Q\in\mathcal{D}}\frac{1}{\left\vert Q\right\vert}\left\vert\sum_{I\in\mathcal{D}:\ I\neq Q}\left\langle f,h_{I;\kappa}\right\rangle\left\langle h_{I;\kappa},h_{Q;\kappa}^{\eta}\right\rangle\right\vert^{2}\mathbf{1}_{Q}\right)^{\frac{1}{2}}\right\Vert_{L^{p}}\right),$$

where for some $C_p, c_p > 0$,

$$C_p \|f\|_{L^p}^p \geq \left\| \left(\sum_{Q\in\mathcal{D}} |\langle f, h_{Q;\kappa}\rangle|^2 \frac{1}{|Q|}\mathbf{1}_Q \right)^{\frac{1}{2}} \right\|_{L^p}^p = \left\| \left(\sum_{Q\in\mathcal{D}} |\triangle_{Q;\kappa} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p}^p \geq c_p \|f\|_{L^p}^p \ .$$

Thus we have for each $Q \in \mathcal{D}$,

$$\sum_{I\in\mathcal{D}:\ I\neq Q} \langle f, h_{I;\kappa}\rangle \left\langle h_{I;\kappa}, h_{Q;\kappa}^{\eta} \right\rangle = \sum_{\substack{I\in\mathcal{D}:\ \ell(I)<\ell(Q)\\ I\in\mathrm{Car}(Q)}} \langle f, h_{I;\kappa}\rangle \left\langle h_{I;\kappa}, h_{Q;\kappa}^{\eta} \right\rangle + \sum_{\substack{I\in\mathcal{D}:\ \ell(I)>\ell(Q)\\ Q\cap\mathcal{H}_{\frac{\eta}{2}}(I)\neq\emptyset}} \langle f, h_{I;\kappa}\rangle \left\langle h_{I;\kappa}, h_{Q;\kappa}^{\eta} \right\rangle$$
$$+ \sum_{\substack{I\in\mathcal{D}:\ \ell(I)\geq\ell(Q)\geq\eta\ell(I)\\ Q\in\mathrm{Car}(I)}} \langle f, h_{I;\kappa}\rangle \left\langle h_{I;\kappa}, h_{Q;\kappa}^{\eta} \right\rangle .$$

As a consequence of the estimates in (2.13), we have for each $Q \in \mathcal{D}$,

$$\begin{aligned}\left| \sum_{I\in\mathcal{D}:\ I\neq Q} \langle f, h_{I;\kappa}\rangle \left\langle h_{I;\kappa}, h_{Q;\kappa}^{\eta} \right\rangle \right| &\lesssim \left| \sum_{\substack{I\in\mathcal{D}:\ \ell(I)<\ell(Q)\\ I\in\mathrm{Car}(Q)}} \langle f, h_{I;\kappa}\rangle \left\langle h_{I;\kappa}, h_{Q;\kappa}^{\eta} \right\rangle \right| + \sum_{\substack{I\in\mathcal{D}:\ \ell(Q)\leq\eta\ell(I)\\ Q\cap\mathcal{H}_{\frac{\eta}{2}}(I)\neq\emptyset}} |\langle f, h_{I;\kappa}\rangle| \frac{1}{\eta^{\kappa}} \left(\frac{\ell(Q)}{\ell(I)} \right)^{\kappa+\frac{n}{2}} \\ &\quad + \eta \sum_{\substack{I\in\mathcal{D}:\ \ell(I)\geq\ell(Q)\geq\eta\ell(I)\\ Q\in\mathrm{Car}(I)}} |\langle f, h_{I;\kappa}\rangle| \left(\frac{\ell(Q)}{\ell(I)} \right)^{\frac{n}{2}} \\ &\equiv A(Q) + B(Q) + C(Q) .\end{aligned}$$

Altogether we have

$$\left\| \left(\sum_{Q\in\mathcal{D}} \frac{1}{|Q|} \left| \sum_{I\in\mathcal{D}:\ I\neq Q} \langle f, h_{I;\kappa}\rangle \left\langle h_{I;\kappa}, h_{Q;\kappa}^{\eta} \right\rangle \right|^2 \mathbf{1}_Q \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \left\| \left(\sum_{Q\in\mathcal{D}} \frac{1}{|Q|} A(Q)^2 \mathbf{1}_Q \right)^{\frac{1}{2}} \right\|_{L^p}$$
$$+ \left\| \left(\sum_{Q\in\mathcal{D}} \frac{1}{|Q|} B(Q)^2 \mathbf{1}_Q \right)^{\frac{1}{2}} \right\|_{L^p} + \left\| \left(\sum_{Q\in\mathcal{D}} \frac{1}{|Q|} C(Q)^2 \mathbf{1}_Q \right)^{\frac{1}{2}} \right\|_{L^p} . \tag{2.14}$$

We now claim that

$$\left\| \left(\sum_{Q\in\mathcal{D}} \frac{1}{|Q|} \left| \sum_{I\in\mathcal{D}:\ I\neq Q} \langle f, h_{I;\kappa}\rangle \left\langle h_{I;\kappa}, h_{Q;\kappa}^{\eta} \right\rangle \right|^2 \mathbf{1}_Q \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \eta^{\frac{1}{2}\gamma_p} \left(\log_2 \frac{1}{\eta} \right) \|f\|_{L^p} . \tag{2.15}$$

With this established, and taking $\kappa > \frac{n}{2}$, we obtain just as in the proof of Proposition 16,

$$\left\| \left(\sum_{Q\in\mathcal{D}} \frac{1}{|Q|} \left| \sum_{I\in\mathcal{D}:\ I\neq Q} \langle f, h_{I;\kappa}\rangle \left\langle h_{I;\kappa}, h_{Q;\kappa}^{\eta} \right\rangle \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \leq C\eta^{\frac{1}{2}\gamma_p} \left(\log_2 \frac{1}{\eta} \right) \|f\|_{L^p} < \frac{c_p}{2} \|f\|_{L^p} \ ,$$

with $\eta > 0$ sufficiently small. This then gives

$$C_p \|f\|_{L^p} \geq \left\| \left(S_{\eta}^{\mathcal{D}} \right)^{\mathrm{tr}} f \right\|_{L^p} \geq c_p \|f\|_{L^p} - \frac{c_p}{2} \|f\|_{L^p} = \frac{c_p}{2} \|f\|_{L^p} \ ,$$

which completes the proof of Proposition 17 modulo (2.15).

We prove (2.15) by estimating each of the three terms on the right hand side of (2.14) separately. These three terms are handled exactly as in Proposition 16 except that the arguments for handling terms A and C are switched, with term B handled the same as before. We leave the routine verifications to the reader, and this finishes our proof of Proposition 17. □

2.3.3. *Representation.* Combining the two propositions above immediately gives the proof of Theorem 14, as we now show.

Proof of Theorem 14. Fix a grid $\mathcal{D}$ in $\mathbb{R}^n$. Combining the two propositions shows that $S_\eta^{\mathcal{D}}$ is a bounded invertible linear map on L^p. Indeed, Proposition 16 shows that $S_\eta^{\mathcal{D}}$ is one-to-one and Proposition 17 shows that $S_\eta^{\mathcal{D}}$ is onto. The boundedness of $S_\eta^{\mathcal{D}}$ is immediate from Proposition 16 and the boundedness of $\left(S_\eta^{\mathcal{D}}\right)^{-1}$ now follows from the Open Mapping Theorem.

Thus dropping the superscript $\mathcal{D}$ we have

$$f = S_\eta \left(S_\eta\right)^{-1} f = \sum_{I\in\mathcal{D}} \left\langle \left(S_\eta\right)^{-1} f, h_{I;\kappa} \right\rangle h_{I;\kappa}^\eta \ .$$

If we set

$$\widetilde{\triangle}_I^\eta f \equiv \left\langle S_\eta^{-1} f, h_{I;\kappa}\right\rangle h_{I;\kappa}^\eta = \triangle_I^\eta \left(S_\eta^{-1} f\right) = \left\langle S_\eta^{-1} f, h_{I;\kappa}\right\rangle \left(\phi_{\eta\ell(I)} * h_{I;\kappa}\right),$$

then we have

$$\begin{aligned} f &= \sum_{I\in\mathcal{D}} \widetilde{\triangle}_I^\eta f = \sum_{I\in\mathcal{D}} \left\langle S_\eta^{-1} f, h_{I;\kappa}\right\rangle h_{I;\kappa}^\eta, \qquad \text{for } f\in L^p, \\ \left\| \left(\sum_{I\in\mathcal{D}} \left| \widetilde{\triangle}_I^\eta f\right|^2 \right)^{\frac12} \right\|_{L^p(\sigma)} &\approx \left\| \left(\sum_{I\in\mathcal{D}} \left|\left\langle S_\eta^{-1} f, h_{I;\kappa}\right\rangle\right|^2 \frac{1}{|I|_\sigma} \mathbf{1}_I \right)^{\frac12} \right\|_{L^p(\sigma)} \approx \left\| S_\eta^{-1} f\right\|_{L^p(\sigma)} \approx \|f\|_{L^p(\sigma)} \ , \\ \left\| \left(\sum_{I\in\mathcal{D}} \left| \triangle_I^\eta f\right|^2 \right)^{\frac12} \right\|_{L^p(\sigma)} &\approx \left\| \left(\sum_{I\in\mathcal{D}} \left|\left\langle f, h_{I;\kappa}\right\rangle\right|^2 \frac{1}{|I|_\sigma} \mathbf{1}_I \right)^{\frac12} \right\|_{L^p(\sigma)} \approx \|f\|_{L^p(\sigma)} \ , \end{aligned}$$

which shows in particular that $\left\{\widetilde{\triangle}_{I;\kappa}^\eta\right\}_{I\in\mathcal{D}}$ is a frame for L^p. □

Notation 20. *Since the frame* $\left\{\widetilde{\triangle}_{I;\kappa}^\eta\right\}_{I\in\mathcal{D}}$ *will be used extensively in what follows, we drop the tilde and write* $\triangle_{I;\kappa}^\eta$ *instead of* $\widetilde{\triangle}_{I;\kappa}^\eta$, *i.e. we* redefine $\triangle_{I;\kappa}^\eta f$ *to be*

$$\triangle_I^\eta f \equiv \sum_{I\in\mathcal{D}} \left\langle S_\eta^{-1} f, h_{I;\kappa}\right\rangle h_{I;\kappa}^\eta,$$

as was done in the Introduction. Thus we have inserted the bounded invertible operator S_η^{-1} *into the inner product above.*

2.3.4. *The smoothed pseudoprojections.* The smoothed operators $\triangle_{I;\kappa}^\eta$ are neither self-adjoint, projections nor orthogonal, but come close as we now show. Recall that

$$\triangle_{I;\kappa}^\eta f = \left\langle \left(S_{\kappa,\eta}\right)^{-1} f, h_{I;\kappa}\right\rangle h_{I;\kappa}^\eta \ , \qquad \text{where } h_{I;\kappa}^\eta = \phi_\eta * h_{I;\kappa} \ .$$

Lemma 21. *With notation as above and* $\phi = \phi_0 * \phi_0$, *we have*

$$\left(\triangle_{I;\kappa}^\eta\right)^{\mathrm{tr}} g = \left\langle g, h_{I;\kappa}^\eta\right\rangle \left(\left(S_{\kappa,\eta}\right)^{-1}\right)^{\mathrm{tr}} h_{I;\kappa} \ ,$$

and

$$\left(\triangle_{I;\kappa}^\eta\right)^2 = a_{I;\kappa}^\eta \triangle_{I;\kappa}^\eta \text{ and } \left[\left(\triangle_{I;\kappa}^\eta\right)^{\mathrm{tr}}\right]^2 = a_{I;\kappa}^\eta \left(\triangle_{I;\kappa}^\eta\right)^{\mathrm{tr}} \text{ and } \left(\triangle_{I;\kappa}^\eta\right)\left(\triangle_{I;\kappa}^\eta\right)^{\mathrm{tr}} = b_{I;\kappa}^\eta \widetilde{\triangle}_{I;\kappa}^\eta = b_{I;\kappa}^\eta \widetilde{\triangle}_{I;\kappa}^\eta,$$

where $\widetilde{\triangle}_{I;\kappa}^\eta f = \left\langle f, h_{I;\kappa}^\eta\right\rangle h_{I;\kappa}^\eta$, *and*

where $a_{I;\kappa}^\eta \equiv \left\langle \left(S_{\kappa,\eta}\right)^{-1} h_{I;\kappa}^\eta, h_{I;\kappa}\right\rangle \approx 1$ *and* $b_{I;\kappa}^\eta \equiv \left\langle \left(S_{\kappa,\eta}\right)^{-2} h_{I;\kappa}, h_{I;\kappa}\right\rangle \approx 1.$

In particular we have

$$\|f\|_{L^p} \approx \left\| \left(\sum_{I\in\mathcal{D}} \frac{\left|\left\langle f, h_{I;\kappa}^\eta\right\rangle\right|^2}{|I|} \mathbf{1}_I \right)^{\frac12} \right\|_{L^p} .$$

Proof. The adjoint property follows from

$$\begin{aligned}\left\langle \triangle_{I;\kappa}^{\eta} f,g\right\rangle &= \left\langle \left\langle \left(S_{\kappa,\eta}\right)^{-1} f,h_{I;\kappa}\right\rangle h_{I;\kappa}^{\eta},g\right\rangle =\left\langle h_{I;\kappa}^{\eta},g\right\rangle \int \left(S_{\kappa,\eta}\right)^{-1} f\left(x\right)h_{I;\kappa}\left(x\right)dx\\ &= \left\langle h_{I;\kappa}^{\eta},g\right\rangle \int f\left(x\right)\left(\left(S_{\kappa,\eta}\right)^{-1}\right)^{\mathrm{tr}} h_{I;\kappa}\left(x\right)dx\\ &= \int f\left(x\right)\left\{\left(\left(S_{\kappa,\eta}\right)^{-1}\right)^{\mathrm{tr}} h_{I;\kappa}\left(x\right)\left\langle h_{I;\kappa}^{\eta},g\right\rangle\right\} dx=\left\langle f,\left(\triangle_{I;\kappa}^{\eta}\right)^{\mathrm{tr}} g\right\rangle .\end{aligned}$$

The pseudoprojection property follows from

$$\begin{aligned}&\left(\triangle_{I;\kappa}^{\eta}\right)^{2} f=\triangle_{I;\kappa}^{\eta}\left(\triangle_{I;\kappa}^{\eta} f\right)=\left\langle \left(S_{\kappa,\eta}\right)^{-1}\left(\triangle_{I;\kappa}^{\eta} f\right),h_{I;\kappa}\right\rangle h_{I;\kappa}^{\eta}\\ =&\left\langle \left(S_{\kappa,\eta}\right)^{-1}\left\{\left\langle \left(S_{\kappa,\eta}\right)^{-1} f,h_{I;\kappa}\right\rangle h_{I;\kappa}^{\eta}\right\},h_{I;\kappa}\right\rangle h_{I;\kappa}^{\eta}=\left\langle \left(S_{\kappa,\eta}\right)^{-1} f,h_{I;\kappa}\right\rangle \left\langle \left(S_{\kappa,\eta}\right)^{-1} h_{I;\kappa}^{\eta},h_{I;\kappa}\right\rangle h_{I;\kappa}^{\eta}\\ =&\left\langle \left(S_{\kappa,\eta}\right)^{-1} h_{I;\kappa}^{\eta},h_{I;\kappa}\right\rangle \left\langle \left(S_{\kappa,\eta}\right)^{-1} f,h_{I;\kappa}\right\rangle h_{I;\kappa}^{\eta}=\left\langle \left(S_{\kappa,\eta}\right)^{-1} h_{I;\kappa}^{\eta},h_{I;\kappa}\right\rangle_{\mu} \triangle_{I;\kappa}^{\eta} f=a_{I;\kappa}^{\eta}\triangle_{I;\kappa}^{\eta} f.\end{aligned}$$

However, $\left(S_{\kappa,\eta}\right)^{-1}$ is close to the identity map by (1.14), so that using $\phi_{\eta}=\phi_{\eta_0}*\phi_{\eta_0}$, we obtain

$$\begin{aligned}a_{I;\kappa}^{\eta} &= \left\langle \left(S_{\kappa,\eta}\right)^{-1} h_{I;\kappa}^{\eta},h_{I;\kappa}\right\rangle \approx \left\langle h_{I;\kappa}^{\eta},h_{I;\kappa}\right\rangle +o\left(1\right)=\left\langle \phi_{\eta\ell(I)}*h_{I;\kappa},h_{I;\kappa}\right\rangle +o\left(1\right)\\ &= \left\langle \phi_{\eta_0\ell(I)}*h_{I;\kappa},\phi_{\eta_0\ell(I)}*h_{I;\kappa}\right\rangle +o\left(1\right)=\left\|h_{I;\kappa}^{\eta_0}\right\|_{L^2}^{2}+o\left(1\right)\approx \left\|h_{I;\kappa}\right\|_{L^2}^{2}+o\left(1\right)\approx 1.\end{aligned}$$

We also compute

$$\begin{aligned}&\left(\triangle_{I;\kappa}^{\eta}\right)\left(\triangle_{I;\kappa}^{\eta}\right)^{\mathrm{tr}} f=\left\langle \left(S_{\kappa,\eta}\right)^{-1}\left(\triangle_{I;\kappa}^{\eta}\right)^{\mathrm{tr}} f,h_{I;\kappa}\right\rangle h_{I;\kappa}^{\eta}\\ =&\left\langle \left(S_{\kappa,\eta}\right)^{-1}\left\{\left\langle f,h_{I;\kappa}^{\eta}\right\rangle \left(S_{\kappa,\eta}\right)^{-1} h_{I;\kappa}\right\},h_{I;\kappa}\right\rangle h_{I;\kappa}^{\eta}=\left\langle f,h_{I;\kappa}^{\eta}\right\rangle \left\langle \left(S_{\kappa,\eta}\right)^{-2} h_{I;\kappa},h_{I;\kappa}\right\rangle h_{I;\kappa}^{\eta}\\ =&\left\langle \left(S_{\kappa,\eta}\right)^{-2} h_{I;\kappa},h_{I;\kappa}\right\rangle \left\langle f,h_{I;\kappa}^{\eta}\right\rangle h_{I;\kappa}^{\eta}=\left\langle \left(S_{\kappa,\eta}\right)^{-2} h_{I;\kappa},h_{I;\kappa}\right\rangle \widetilde{\triangle}_{I;\kappa}^{\eta} f.\end{aligned}$$

Finally,

$$f=\sum_{I\in\mathcal{D}}\left(\triangle_{I;\kappa}^{\eta}\right)^{\mathrm{tr}} f=\sum_{I\in\mathcal{D}}\left\langle f,h_{I;\kappa}^{\eta}\right\rangle \left[\left(S_{\kappa,\eta}^{\mathrm{tr}}\right)^{-1}\right]^{\mathrm{tr}} h_{I;\kappa}=\left[\left(S_{\kappa,\eta}^{\mathrm{tr}}\right)^{-1}\right]^{\mathrm{tr}}\sum_{I\in\mathcal{D}}\left\langle f,h_{I;\kappa}^{\eta}\right\rangle h_{I;\kappa}$$

shows that

$$\left\|f\right\|_{L^p}=\left\|\left[\left(S_{\kappa,\eta}^{\mathrm{tr}}\right)^{-1}\right]^{\mathrm{tr}}\sum_{I\in\mathcal{D}}\left\langle f,h_{I;\kappa}^{\eta}\right\rangle h_{I;\kappa}\right\|_{L^p}\approx\left\|\sum_{I\in\mathcal{D}}\left\langle f,h_{I;\kappa}^{\eta}\right\rangle h_{I;\kappa}\right\|_{L^p}\approx\left\|\left(\sum_{I\in\mathcal{D}}\frac{\left|\left\langle f,h_{I;\kappa}^{\eta}\right\rangle\right|^{2}}{\left|I\right|}\mathbf{1}_I\right)^{\frac{1}{2}}\right\|_{L^p}$$

□

3. The extension operator and oscillatory inner products

Given $f\in L^p\left(\sigma_{n-1}\right)$, we define the extension operator E_χ localized to a cutoff function $\chi\left(x\right)$ by

$$E_\chi f\left(\xi\right)=\mathcal{F}\left(f\sigma_{n-1}\right)\left(\xi\right)=\int_{\mathbb{S}^{n-1}} f\left(z\right)e^{-iz\cdot\xi}\chi\left(z\right)d\sigma_{n-1}\left(z\right).$$

If we use a one-to-one onto coordinate patch $\Phi:U\to\mathbb{P}$ such that $\operatorname{Supp}\chi\subset\mathbb{P}$ and U is a cube centered at the origin in $\mathbb{R}^{n-1}$ with dyadic side length, then we can write

$$\begin{aligned}E_\chi f\left(\xi\right) &= \int_{\mathbb{P}} f\left(y\right)e^{-iy\cdot\xi}\chi\left(y\right)d\sigma_{n-1}\left(y\right)=\int_U f\left(\Phi\left(x\right)\right)e^{-i\Phi\left(x\right)\cdot\xi}\chi\left(\Phi\left(x\right)\right)\frac{dx}{\left|\det\nabla\Phi\left(x\right)\right|}\\ &= \int_U h\left(x\right)e^{-i\Phi\left(x\right)\cdot\xi}\zeta\left(x\right)dx\end{aligned}$$

where

$$h\left(x\right)=f\left(\Phi\left(\mathbb{P}x\right)\right)\ \text{ and }\ \zeta\left(x\right)\equiv\frac{\chi\left(\Phi\left(x\right)\right)}{\left|\det\nabla\Phi\left(x\right)\right|}.$$

Since the map $\Phi : U \to \mathbb{P}$ is a diffeomorphism, we have

$$\|h\|_{L^p(U)} \approx \|f\|_{L^p(\mathbb{P})} ,$$

and thus the extension operator $E_\chi : L^p(\sigma_{n-1}) \to L^p(\mathbb{R}^n)$ is bounded if and only if the linear map $T : L^p(U) \to L^p(\mathbb{R}^n)$ is bounded, where T is defined by

$$Tf(\xi) \equiv \int_{B_{n-1}\left(0,\frac{1}{2}\right)} K_{\Phi,\zeta}(x,\xi) f(x)\,dx = \int_{B_{n-1}\left(0,\frac{1}{2}\right)} f(x) e^{-i\Phi(x)\cdot\xi} dx, \qquad \text{for } f \in L^p\left(B_{n-1}\left(0,\frac{1}{2}\right)\right),$$
$$\text{where } K_{\Phi,\zeta}(x,\xi) \equiv e^{-i\Phi(x)\cdot\xi}.$$

Now recall the $(n-1)$-dimensional Alpert wavelets $\left\{h_{I;\kappa}^{n-1}\right\}_{I\in\mathcal{G}}$ on $\mathbb{R}^{n-1}$ where $\mathcal{G}$ is a translation of the standard dyadic grid on $\mathbb{R}^{n-1}$ so that $S \in \mathcal{G}$ and the origin is a vertex of $\pi_{\mathcal{G}}^{(2)} S$ (see also Notation 15), and recall the smooth analogues $h_{I;\kappa}^{n-1,\eta}$ of these wavelets as constructed in Theorem 6 above. Then expand f by the smooth Alpert reproducing formula $f = S_{\kappa,\eta} S_{\kappa,\eta}^{-1} f = \sum_{I\in\mathcal{G}} \left\langle S_{\kappa,\eta}^{-1} f, h_{I;\kappa}^{n-1}\right\rangle h_{I;\kappa}^{n-1,\eta}$. In addition recall the n-dimensional Alpert wavelets $\left\{h_{J;\kappa}^{n}\right\}_{J\in\mathcal{D}}$ on $\mathbb{R}^n$, where $\mathcal{D}$ is the standard grid on $\mathbb{R}^n$, together with their smooth analogues $h_{J;\kappa}^{n,\eta}$. It will be important, at least in a technical sense when estimating part of the above form in Section 7, to use the standard grid $\mathcal{D}$ on $\mathbb{R}^n$ which enjoys the property that the distance from the origin to a cube $J \in \mathcal{D}$ is at least the side length of J, if the origin is not a vertex of J.

To estimate the left hand side $\left\|T \sum_{I\in\mathcal{G}[U]} \triangle_{I;\kappa}^{\eta} f\right\|_{L^p(\lambda_n)}$ of the truncated extension inequality (1.9) when $p = q$, we will use in particular the vanishing moments up to order $\kappa - 1$ of the wavelets $h_{I;\kappa}^{n-1,\eta}$ and $h_{J;\kappa}^{n,\eta}$,

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} h_{I;\kappa}^{n-1,\eta}(x)\, x^\alpha dx &= 0, \qquad \text{for } 0 \le |\alpha| < \kappa, \\ \int_{\mathbb{R}^{n}} h_{J;\kappa}^{n,\eta}(\xi)\, \xi^\alpha d\xi &= 0, \qquad \text{for } 0 \le |\alpha| < \kappa, \end{aligned}$$

along with estimates for oscillatory integrals in which the amplitudes involve smooth Alpert wavelets.

We will now estimate the oscillatory inner product

$$\left\langle T h_{I;\kappa}^{n-1,\eta}, h_{J;\kappa}^{n,\eta}\right\rangle = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{n-1}} e^{-i\Phi(x)\cdot\xi} h_{I;\kappa}^{n-1,\eta}(x)\,dx\right) h_{J;\kappa}^{n,\eta}(\xi)\,d\xi, \tag{3.1}$$

for $(I,J) \in \mathcal{G}[U] \times \mathcal{D}$ and plug the resulting estimates into the decomposition of the pairs (I,J) of dyadic cubes in $\mathcal{P}$ given in (1.22) of the introduction, namely

$$\mathcal{P} = \mathcal{P}_0 \ \cup\ \bigcup_{m=0}^{\infty} \mathcal{P}_m \ \cup\ \mathcal{R} \ \cup\ \mathcal{X} \ .$$

Thus $\mathcal{P}_0$ consists of pairs that are aligned radially away from the origin, $\mathcal{P}_m$ consists of pairs that are radially staggered at angle roughly 2^{-m}, $\mathcal{R}$ consists of pairs where I is 'close' to the larger J, and $\mathcal{X}$ consists of pairs in which the spherical projection of J is disjoint from $\Phi(2U)$.

Regarding $\mathcal{P}_0$, our intuition tells us that when the approximate wavelength $\frac{1}{|\xi|}$ of the exponential $e^{-ix\cdot\xi}$ does not exceed the depth $\frac{1}{\ell(I)^2}$ of the spherical 'cap' $\Phi(I)$, and the side length $\ell(J)$ of the cube J supporting $h_{J;\kappa}^{n,\eta}$ is approximately the distance of the sphere from the origin, namely 1, then we should *not* expect to derive any cancellation from the presence of the exponential $e^{-i\Phi(x)\cdot\xi}$. Thus the only estimate on the inner product in this case should be the trivial one, in which the oscillatory factor $e^{-i\Phi(x)\cdot\xi}$ is discarded,

$$\left|\left\langle T h_{I;\kappa}^{n-1,\eta}, h_{J;\kappa}^{n,\eta}\right\rangle\right| \le \left\|h_{I;\kappa}^{n-1,\eta}\right\|_{L^1} \left\|h_{J;\kappa}^{n,\eta}\right\|_{L^1} . \tag{3.2}$$

While this crude estimate will ultimately prove adequate in the case when $\ell(J) \approx 1$, $\frac{1}{\ell(I)} \lesssim \frac{1}{\operatorname{dist}(0,J)} \approx \frac{1}{|\xi|} \lesssim \frac{1}{\ell(I)^2}$ and I and J are suitably aligned in the same direction, we must obtain improvements with geometric decay in parameters $|k|$ and $d \ge 0$ when

$$\ell(J) = 2^k \text{ and } \frac{2^{d-1}}{\ell(I)^2} \le \operatorname{dist}(0,J) \le \frac{2^{d+1}}{\ell(I)^2} \ell(J) \lesssim 1.$$

Moreover, when I and J are not suitably aligned, and there is insufficient oscillation within the inner product, we will need to invoke interpolation arguments with L^2 and average L^4 estimates when acting on certain Alpert pseudoprojections.

When $k > 0$, we will gain geometically if we integrate by parts radially in ξ using the smoothness of the wavelets $h^{n,\eta}_{J;\kappa}$, and when $k < 0$, we will gain geometrically in $|k|$ using the large number of vanishing moments of $h^{n,\eta}_{J;\kappa}$. When $d > 0$, we will use the classical asymptotic formula for the smooth surface carried measure $h^{n-1,\eta}_{I;\kappa}$ with sharp bounds on the derivatives of $h^{n-1,\eta}_{I;\kappa}$ to derive gain. Regarding $\mathcal{P}_m$, we will use in addition a tangential integration by parts decay principle since the critical point of the phase no longer lies in the support of the amplitude (hence stationary phase is not needed here). This suggests that we further decompose the index set $\mathcal{P}_0$ as

$$\begin{aligned} \mathcal{P}_0 &= \bigcup_{k\in Z}\bigcup_{d=1}^{\infty} \mathcal{P}_0^{k,d}, \text{ where} \\ \mathcal{P}_0^{k,d} &\equiv \left\{ (I,J)\in\mathcal{P} : J\subset\mathcal{K}(I),\, \ell(J)=2^k, \text{ and } \frac{2^{d-1}}{\ell(I)^2} \le \operatorname{dist}(0,J) = \frac{2^{d+1}}{\ell(I)^2} \right\}, \end{aligned} \tag{3.3}$$

for $k, d \in \mathbb{Z}$, and the index set $\mathcal{P}_m$ of pairs as

$$\begin{aligned} &\mathcal{P}_m = \bigcup_{k\in\mathbb{Z}}\bigcup_{d\in\mathbb{Z}}^{\infty} \mathcal{P}_m^{k,d}, \text{ where} \\ &\mathcal{P}_m^{k,d} \equiv \left\{ (I,J)\in\mathcal{P}_m : 2^{m+1}I\subset U,\, \ell(J)=2^k, \text{ and } 2^d \le \ell(I)^2 \operatorname{dist}(0,J) \le 2^{d+1} \right\}, \end{aligned} \tag{3.4}$$

for $k, d \in \mathbb{Z}$ and $m \in \mathbb{N}$. For $m \in \mathbb{N}$ and $d \le 0$, a different pigeonholing that respects resonance is required, which we defer until needed in Section 8. Similarly, we defer further pigeonholing of $\mathcal{R}$ and $\mathcal{X}$ until needed. In all of these index sets, the cubes I are restricted to $\mathcal{G}[U]$.

Next we introduce a standard change of variable that simplifies calculations, and then derive the well-known asymptotic formula we will use with estimates on the remainder term[8].

3.1. A change of variables.

Write $z = (z', z_n)$ for $z \in \mathbb{R}^n$, and set

$$\phi(x,y) = \Phi(x)\cdot\Phi(y), \qquad \text{where } \Phi(x) = \left(x, \sqrt{1-|x|^2}\right) \text{ and } x\in\mathbb{R}^{n-1}, \tag{3.5}$$

and define the variables (y, λ) by

$$y = \Phi^{-1}\left(\frac{\xi}{|\xi|}\right) = \frac{\xi'}{|\xi|} \text{ and } \lambda = |\xi|, \qquad \text{i.e. } (\xi',\xi_n) = \xi = \lambda\Phi(y) = \left(\lambda y, \lambda\sqrt{1-|y|^2}\right), \tag{3.6}$$

since then

$$\lambda\phi(x,y) = |\xi|\,\Phi(x)\cdot\Phi(y) = |\xi|\,\Phi(x)\cdot\frac{\xi}{|\xi|} = \Phi(x)\cdot\xi\ .$$

We claim that

$$\det\frac{\partial(\xi',\xi_n)}{\partial(y,\lambda)} = \frac{|\xi|^n}{\xi_n}.$$

[8] These estimates are undoubtedly in the literature, but since the author was unable to find the precise form used here, we include the classical arguments below.

Indeed, we have $(y,\lambda)=\left(\frac{\xi'}{|\xi|},|\xi|\right)$ and $\xi=\lambda\left(y,\sqrt{1-|y|^2}\right)$, and so

$$\frac{\partial\left(y_1,...,y_{n-1},\lambda\right)}{\partial\left(\xi_1,...,\xi_{n-1},\xi_n\right)}=\left[\begin{array}{cccc}\frac{\partial}{\partial\xi_1}\frac{\xi_1}{|\xi|} & \cdots & \frac{\partial}{\partial\xi_{n-1}}\frac{\xi_1}{|\xi|} & \frac{\partial}{\partial\xi_n}\frac{\xi_1}{|\xi|}\\ \vdots & \ddots & \vdots & \vdots\\ \frac{\partial}{\partial\xi_1}\frac{\xi_{n-1}}{|\xi|} & \cdots & \frac{\partial}{\partial\xi_{n-1}}\frac{\xi_{n-1}}{|\xi|} & \frac{\partial}{\partial\xi_n}\frac{\xi_{n-1}}{|\xi|}\\ \frac{\partial}{\partial\xi_1}|\xi| & \cdots & \frac{\partial}{\partial\xi_{n-1}}|\xi| & \frac{\partial}{\partial\xi_n}|\xi|\end{array}\right]$$

$$=\left[\begin{array}{cccc}\frac{1}{|\xi|}-\frac{\xi_1^2}{|\xi|^3} & \cdots & -\frac{\xi_1\xi_{n-1}}{|\xi|^3} & -\frac{\xi_1\xi_n}{|\xi|^3}\\ \vdots & \ddots & \vdots & \vdots\\ -\frac{\xi_1\xi_{n-1}}{|\xi|^3} & \cdots & \frac{1}{|\xi|}-\frac{\xi_{n-1}^2}{|\xi|^3} & -\frac{\xi_{n-1}\xi_n}{|\xi|^3}\\ \frac{\xi_1}{|\xi|} & \cdots & \frac{\xi_{n-1}}{|\xi|} & \frac{\xi_n}{|\xi|}\end{array}\right]=\frac{1}{|\xi|^3}\left[\begin{array}{cccc}|\xi|^2-\xi_1^2 & \cdots & -\xi_1\xi_{n-1} & -\xi_1\xi_n\\ \vdots & \ddots & \vdots & \vdots\\ -\xi_1\xi_{n-1} & \cdots & |\xi|^2-\xi_{n-1}^2 & -\xi_{n-1}\xi_n\\ \xi_1|\xi|^2 & \cdots & \xi_{n-1}|\xi|^2 & \xi_n|\xi|^2\end{array}\right]$$

where

$$\det\left[\begin{array}{cccc}|\xi|^2-\xi_1^2 & \cdots & -\xi_1\xi_{n-1} & -\xi_1\xi_n\\ \vdots & \ddots & \vdots & \vdots\\ -\xi_1\xi_{n-1} & \cdots & |\xi|^2-\xi_{n-1}^2 & -\xi_{n-1}\xi_n\\ \xi_1|\xi|^2 & \cdots & \xi_{n-1}|\xi|^2 & \xi_n|\xi|^2\end{array}\right]$$

$$=|\xi|^2\det\left[\begin{array}{cccc}|\xi|^2-\xi_1^2 & \cdots & -\xi_1\xi_{n-1} & -\xi_1\xi_n\\ \vdots & \ddots & \vdots & \vdots\\ -\xi_1\xi_{n-1} & \cdots & |\xi|^2-\xi_{n-1}^2 & -\xi_{n-1}\xi_n\\ \xi_1 & \cdots & \xi_{n-1} & \xi_n\end{array}\right]=|\xi|^2\xi_n|\xi|^{2(n-1)}=\xi_n|\xi|^{2n},$$

by an induction on $n\in\mathbb{N}$.

Thus we have

$$\begin{aligned}\det\frac{\partial\left(y_1,...,y_{n-1},\lambda\right)}{\partial\left(\xi_1,...,\xi_{n-1},\xi_n\right)}&=\frac{1}{|\xi|^{3n}}\det\left[\begin{array}{cccc}|\xi|^2-\xi_1^2 & \cdots & -\xi_1\xi_{n-1} & -\xi_1\xi_n\\ \vdots & \ddots & \vdots & \vdots\\ -\xi_1\xi_{n-1} & \cdots & |\xi|^2-\xi_{n-1}^2 & -\xi_{n-1}\xi_n\\ \xi_1|\xi|^2 & \cdots & \xi_{n-1}|\xi|^2 & \xi_n|\xi|^2\end{array}\right]\\ &=\frac{1}{|\xi|^{3n}}\xi_n|\xi|^{2n}=\frac{\xi_n}{|\xi|^n},\end{aligned}$$

as claimed. Hence

$$\det\frac{\partial\left(\xi_1,...,\xi_{n-1},\xi_n\right)}{\partial\left(y_1,...,y_{n-1},\lambda\right)}=\frac{|\xi|^n}{\xi_n}=\frac{\lambda^n}{\lambda\sqrt{1-|y|^2}}=\frac{\lambda^{n-1}}{\sqrt{1-|y|^2}},$$

and the change of variable $\xi\to(y,\lambda)$ gives,

$$\begin{aligned}&\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle=\int_{\mathbb{R}^n}\int_{B_{n-1}\left(0,\frac{1}{2}\right)}e^{i\Phi(x)\cdot\xi}h_{I;\kappa}^{n-1,\eta}(x)\,h_{J;\kappa}^{n,\eta}(\xi)\,dxd\xi\\ =&\int_{\mathbb{R}^n}\int_{B_{n-1}\left(0,\frac{1}{2}\right)}e^{i\Phi(x)\cdot\lambda\left(y,\sqrt{1-|y|^2}\right)}h_{I;\kappa}^{n-1,\eta}(x)\,h_{J;\kappa}^{n,\eta}\left(\lambda\left(y,\sqrt{1-|y|^2}\right)\right)\det\frac{\partial\left(\xi_1,...,\xi_{n-1},\xi_n\right)}{\partial\left(y_1,...,y_{n-1},\lambda\right)}dxdyd\lambda\\ =&\int_{\mathbb{R}}\int_{B_{n-1}\left(0,\frac{1}{2}\right)}\int_{B_{n-1}\left(0,\frac{1}{2}\right)}e^{i\lambda\Phi(x)\cdot\Phi(y)}h_{I;\kappa}^{n-1,\eta}(x)\,h_{J;\kappa}^{n,\eta}\left(\lambda y,\lambda\sqrt{1-|y|^2}\right)\frac{\lambda^n}{\lambda\sqrt{1-|y|^2}}dxdyd\lambda\\ =&\int_{\mathbb{R}}\int_{B_{n-1}\left(0,\frac{1}{2}\right)}\int_{B_{n-1}\left(0,\frac{1}{2}\right)}e^{i\lambda\phi(x,y)}\varphi_I^\eta(x)\,\widetilde{\psi}_J^\eta(y,\lambda)\,dxdyd\lambda,\end{aligned}$$

where we are now using the convenient notation,

$$\begin{aligned}\phi(x,y) &\equiv \Phi(x)\cdot\Phi(y),\\ \varphi_I^\eta(x) &\equiv h_{I;\kappa}^{n-1,\eta}(x) \text{ and } \psi_J^\eta(\xi)=h_{J;\kappa}^{n,\eta}(\xi),\\ \widetilde{\psi}_J^\eta(y,\lambda) &\equiv h_{J;\kappa}^{n,\eta}\left(\lambda y,\lambda\sqrt{1-|y|^2}\right)\frac{\lambda^{n-1}}{\sqrt{1-|y|^2}}.\end{aligned} \tag{3.7}$$

Note that if $\xi\in J$, then $(y,\lambda)\in\pi_{\tan}J\times\pi_{\mathrm{rad}}J$.

3.2. **Bounds for oscillatory integrals.** Here we review the well known asymptotics for oscillatory integrals, see e.g. [Ste2, Chapter VIII], paying close attention to the constants involved. We emphasize that the results in this subsection are well known, but as we could not find in the literature the exact form of the estimate for the remainder term that we use here, we reproduce many familiar arguments below.

We consider the oscillatory function $\mathcal{I}_{a_\lambda,\phi}:\mathbb{R}^d\times(0,\infty)\to\mathbb{C}$ given by

$$\mathcal{I}_{a_\lambda,\phi}(y,\lambda)\equiv\int_{\mathbb{R}^n}e^{i\lambda\phi(x,y)}a_\lambda(x,y)\,dx,$$

defined for $\lambda>0$ and $y\in U$ where U is an open subset of $\mathbb{R}^d$, and we call $\phi(x,y)$ the phase and $a_\lambda(x,y)$ the amplitude of $\mathcal{I}_{a_\lambda,\phi}$. We will follow a treatment of asymptotics for such oscillatory integrals given in a Rice University blog [blogs.rice], but we will obtain a sharp estimate for amplitudes of the type that arise in the smooth Alpert expansions.

We use three familiar preparatory lemmas. The first of these is the Morse Lemma, which will be applied to the phase function $\phi(x,y)$, in order to transform ϕ into a nonsingular quadratic form in x at a nondegenerate critical point in x. The second lemma gives high order decay bounds in the special case when there are no critical points in x of the phase function that lie in the support of the amplitude, and the third calculates the oscillatory integral for a quadratic form.

Lemma 22 (Morse Lemma)**.** *Suppose $y_0\in U\subset\mathbb{R}^d$ and x_0 is a nondegenerate stationary point for $\phi(\cdot_x,y_0)$. Then there exists a neighbourhood $V\subset U$ of y_0, a neighbourhood W of x_0, a smooth function*

$$X:V\to W,$$

and a smooth function

$$\Psi:V\to W\times V\to\mathbb{R}^n,$$

such that

(1) *For every $y\in V$, $X(y)$ is the unique stationary point, which is also nondegenerate, for $\phi(\cdot_x,y_0)$ in W.*
(2) *For every $y\in V$, the map $W\to\mathbb{R}^n$ defined by $x\to\Psi(x,y)$ is a diffeomorphism onto its image and*

$$\phi(x,y)=\phi(X(y),y)+\frac{1}{2}\Psi(x,y)^{\mathrm{tr}}\left[\partial_x^2\phi(X(y),y)\right]\Psi(x,y). \tag{3.8}$$

Furthermore,

$$\Psi(X(y),y)=0 \text{ and } \partial_x\Psi(X(y),y)=\mathrm{Id}_n. \tag{3.9}$$

(3) *Finally, we may take $W=B(x_0,a\gamma)$ for some small positive constant*

$$a=\frac{c_n}{\max_{|\alpha|\leq 3}\sup_{(x,y)\in(\mathrm{Supp}\,a)\times U}|\partial_x^\alpha\phi(x,y)|},$$

where $\gamma>0$ satisfies $\inf_y\left[\partial_x^2\phi(X(y),y)\right]\succcurlyeq\gamma\,\mathrm{Id}_n$.

Proof. For any y, the stationary points are the solutions of the equation $0=\partial_x\phi(x,y)$, and by the nondegeneracy of the critical point, and the Implicit Function Theorem, this equation uniquely defines x as a function of y in some neighbourhood $\mathcal{N}$ of (x_0,y_0). Since in our application, $\phi(x,y)$ is homogeneous of degree zero in y, we may assume this here as well. Then $\left[\partial_x^2\phi(X(y),y)\right]\succcurlyeq\gamma\,\mathrm{Id}_{n-1}$ for some $\gamma>0$ depending only on ϕ, and so we may take $\mathcal{N}=B((x_0,y_0),a'\gamma)$ where $a'=\frac{c_n'}{\max_{|\alpha|\leq 3}\sup_{(x,y)}|\partial_x^\alpha\phi(x,y)|}$ for some small positive constant c_n', depending only on the dimension n.

Now we take the Taylor expansion of $\phi(x,y)$ in x about $X(y)$ to obtain, upon noting that the first derivatives in the Taylor expansion vanish at the critical point $X(y)$,

$$\phi(x,y) = \phi(X(y),y) + \frac{1}{2}(x - X(y))^{\mathrm{tr}} B(x,y)(x - X(y)),$$
$$\text{where } B(x,y) \equiv \int_0^1 (1-s)\,\partial_x^2\phi(sx + (1-s)X(y),y)\,ds.$$

We now construct a matrix-valued function $R(x,y)$ such that

$$\Psi(x,y) \equiv R(x,y)(x - X(y))$$

has the properties listed in (2) above. Indeed, this Ψ will satisfy (3.8) provided

$$R(x,y)^{\mathrm{tr}}\,\partial_x^2\phi(X(y),y)\,R(x,y) - B(x,y) = 0, \qquad \text{for } (x,y)\in\mathcal{N}. \tag{3.10}$$

We interpret the left hand side of (3.10) as a mapping from $\mathcal{M}_n(\mathbb{R})_R \times \mathbb{R}_x^n \times V_y$ to $\mathcal{S}_n(\mathbb{R})$, where $\mathcal{M}_n(\mathbb{R})$ is the set of $n\times n$ matrices and $\mathcal{S}_n(\mathbb{R})$ is the subset of symmetric matrices. Taking the differential of the left hand side of (3.10) with respect to the variable R and evaluated at the identity matrix Id_n, we obtain that the derivative map,

$$dR \to (dR)^{\mathrm{tr}}\,\partial_x^2\phi(X(y),y) + \partial_x^2\phi(X(y),y)(dR),$$

is surjective, since whenever $C\in\mathcal{S}_n(\mathbb{R})$ is symmetric,

$$\left(\frac{1}{2}\left[\partial_x^2\phi(X(y),y)\right]^{-1}C\right)^{\mathrm{tr}}\partial_x^2\phi(X(y),y) + \partial_x^2\phi(X(y),y)\left(\frac{1}{2}\left[\partial_x^2\phi(X(y),y)\right]^{-1}C\right)$$
$$= \frac{1}{2}C + \frac{1}{2}C = C.$$

Thus by the Implicit Function Theorem again, there exists a smooth $\mathcal{M}_n(\mathbb{R})$-valued function $R(x,y)$ defined on some neighbourhood $\mathcal{N}_0\subset\mathcal{N}$ of (x_0,y_0) that satisfies (3.10) everywhere that it is defined. Note that we may take $\mathcal{N}_0 = B((x_0,y_0),a''\gamma)$ where where $a'' = \frac{c_n''}{\max_{|\alpha|\le 3}\sup_{(x,y)}|\partial_x^\alpha\phi(x,y)|}$. Possibly shrinking even more the neighbourhood $\mathcal{N}_0$ to $\mathcal{N}_1$, completes the proof that there is a neighbourhood W of x_0 such that $x\to\Psi(x,y)$ is a diffeomorphism from W onto its image, and that (3.8) holds, and that $\Psi(X(y),y)=0$. Note that we may take $W = B(x_0,a\gamma)$ where $a = \frac{c_n}{\max_{|\alpha|\le 3}\sup_{(x,y)}|\partial_x^\alpha\phi(x,y)|}$. The remaining assertion $\partial_x\Psi(X(y),y) = \mathrm{Id}_n$ is straightforward since,

$$\partial_x\mid_{x=X(y)}\Psi(X(y),y) = \left[\partial_x R(x,y)(x - X(y)) + R(x,y)\right]\mid_{x=X(y)} = R(X(y),y) = \mathrm{Id}_n\ ,$$

because we evaluated the differential in R of the left hand side of (3.10) at the identity matrix Id_n. □

Recall that

$$\mathcal{I}_{a_\lambda,\phi}(y,\lambda) \equiv \int_{\mathbb{R}^n} e^{i\lambda\phi(x,y)}a_\lambda(x,y)\,dx,$$

where $\phi\in C^\infty(\mathbb{R}_x^n\times U_y)$ and $a_\lambda\in C^\infty(\mathbb{R}_x^n\times U_y)$. We will need the following estimate in the absence of critical points for $x\to\phi(x,y)$.

Lemma 23. *Suppose that the $\mathbb{R}^n$-valued function $\partial_x\phi(x,y)$ is nonvanishing on $(\operatorname{Supp}a)\times U$. Then for every $N\in\mathbb{N}$ and compact $K\Subset U$ we have*

$$\sup_{y\in K}|\mathcal{I}_{a,\phi}(y,\lambda)| \le C_{N,K}\frac{1}{\lambda^N}\sum_{|\alpha|\le N}\sup_{y\in K}\|\partial_x^\alpha a_\lambda\|_{L^1(\mathbb{R}^n)}, \qquad \textit{for } (y,\lambda)\in(\operatorname{Supp}a)\times U.$$

Proof. For any $M\in\mathbb{N}$ we have

$$\mathcal{I}_{a_\lambda,\phi}(y,\lambda) = \int_{\mathbb{R}^n}\frac{\langle\partial_x\phi(x,y),\partial_x\rangle^M e^{i\lambda\phi(x,y)}}{\left(i\lambda|\partial_x\phi(x,y)|^2\right)^M}a_\lambda(x,y)\,dx,$$

and integrating by parts gives

$$\begin{aligned}\sup_{y\in K}\left|\mathcal{I}_{a_\lambda,\phi}(y,\lambda)\right| &\leq \sup_{y\in K}\frac{1}{\lambda^N}\int_{\mathbb{R}^n}\left|\left\langle\partial_x,\frac{\partial_x\phi(x,y)}{\left|\partial_x\phi(x,y)\right|^2}\right\rangle^N a_\lambda(x,y)\right|dx\\ &\leq C_{N,K}\frac{1}{\lambda^N}\sum_{|\alpha|\leq N}\sup_{y\in K}\int_{\mathbb{R}^n}\left|\partial_x^\alpha a_\lambda(x,y)\right|dx\\ &= C_{N,K}\frac{1}{\lambda^N}\sum_{|\alpha|\leq N}\sup_{y\in K}\left\|\partial_x^\alpha a_\lambda\right\|_{L^1(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)}.\end{aligned}$$

□

The final preparatory lemma is the calculation of an oscillatory integral for a quadratic form.

Definition 24. *For a tempered distribution $u\in\mathcal{S}(\mathbb{R}^n)$, we have*

$$\widehat{u}(\xi)=\mathcal{F}(u)(\xi)=\int_{\mathbb{R}^n}e^{-ix\cdot\xi}u(x)\,d(x).$$

Lemma 25. *Let $A\in\mathcal{M}_n(\mathbb{R}^n)$ be symmetric and nondegenerate with signature* $\operatorname{sgn}(A)$*. Then the tempered distribution $e^{ix^{\mathrm{tr}}Ax}$ has Fourier transform given by,*

$$\mathcal{F}\left(e^{ix^{\mathrm{tr}}Ax}\right)(\xi)=\pi^{\frac{n}{2}}e^{i\operatorname{sgn}(A)\frac{\pi}{4}}\frac{e^{-i\frac{\xi^{\mathrm{tr}}A^{-1}\xi}{4}}}{\sqrt{\det(A)}}. \tag{3.11}$$

Proof. The Fourier transform of a Gaussian function $e^{-t|x|^2}$ is given by

$$\mathcal{F}\left(e^{-t|x|^2}\right)(\xi)=\pi^{\frac{n}{2}}\frac{e^{-\frac{|\xi|^2}{4t}}}{t^{\frac{n}{2}}},\qquad\text{for all } t>0.$$

Now note that both sides of the above identity extend to analytic functions of t in the right half plane $\{t\in\mathbb{C}:\operatorname{Re}t>0\}$. A standard limiting argument and orthogonal change of variables gives the formula (3.11). □

3.3. **The main oscillatory integral bound.** Here is the main oscillatory integral bound.

Remark 26. *In the application of stationary phase to bound the below form in Section 6, we won't actually use the oscillatory term $e^{i\lambda\phi(X(y),y)}$ in the asymptotic formula below, and instead we only need the estimates of the modulus of $\mathcal{I}_{a_\lambda,\phi}(y,\lambda)$ that follow from the asymptotic formula using $\left|e^{i\lambda\phi(X(y),y)}\right|=1$. The reason for this is that when dealing with the below subform $\mathsf{B}^{k,d}_{\mathrm{below}}(f,g)$ with $k,d\geq 0$ large, we can* first *apply radial integration by parts in the inner product, and* second *apply stationary phase to the resulting inner product with a new amplitude. This way the geometric gain in k has been achieved without using the oscillatory term $e^{i\lambda\phi(X(y),y)}$. If we were to instead apply stationary phase first, then we would need $e^{i\lambda\phi(X(y),y)}$ for integration by parts afterward.*

Remark 27. *We will only use the case $M=0$ of Theorem 28 in the proof of the probabilistic extension conjecture in Theorem 4, which corresponds to the classical asymptotic formula with just the principal term and remainder, but with a sharp estimate here on the remainder term when the amplitude is a smooth Alpert wavelet.*

We now give a more general treatment of stationary phase than we need, which might be of use elsewhere.

Theorem 28. *Suppose that $a_\lambda(x,y)\in C_c^\infty\left(\mathbb{R}^n_x\times\mathbb{R}^d_y\right)$, $y_0\in U\subset\mathbb{R}^d$, and that $\phi(\cdot_x,y_0)$ has exactly one nondegenerate stationary point on the support of a at x_0. Take V, W, X and Ψ as in the Morse Lemma. Then for every $M\in\mathbb{N}$, there is a positive constant C_M depending on M and ϕ such that,*

$$\mathcal{I}_{a_\lambda,\phi}(y,\lambda)=\mathfrak{P}_{a_\lambda,\phi}(y,\lambda)+\sum_{\ell=1}^{M}\mathfrak{P}^{(\ell)}_{a_\lambda,\phi}(y,\lambda)+\mathfrak{R}^{(M+1)}_{a_\lambda,\phi}(y,\lambda),$$

where

$$\mathfrak{P}_{a_\lambda,\phi}(y,\lambda)=\left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}}\frac{e^{i\left[\operatorname{sgn}\left[\partial_x^2\phi(X(y),y)\right]\frac{\pi}{4}+\lambda\phi(X(y),y)\right]}}{\sqrt{\left|\partial_x^2\phi\left(X\left(y\right),y\right)\right|}}a_\lambda\left(X\left(y\right),y\right),$$

$$\mathfrak{P}^{(\ell)}_{a_\lambda,\phi}(y,\lambda)=\frac{i^\ell}{\left(2\lambda\right)^\ell \ell!}\left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}}\frac{e^{i\left[\operatorname{sgn}B(y)\frac{\pi}{4}+\lambda\phi(X(y),y)\right]}}{\sqrt{\det B\left(y\right)}}$$
$$\times\left\{\left[\partial_x\frac{1}{\det\partial_x\Psi\left(x,y\right)}\right]B\left(y\right)^{-1}\frac{1}{\det\partial_x\Psi\left(x,y\right)}\partial_x\right\}^\ell\frac{a_\lambda\left(x,y\right)}{\det\left[\partial_x\Psi\left(x,y\right)\right]}\ |_{x=X(y)},$$

and

$$\mathfrak{R}^{(M+1)}_{a_\lambda,\phi}(y,\lambda)=\left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}}\frac{e^{i\left[\operatorname{sgn}B(y)\frac{\pi}{4}+\lambda\phi(X(y),y)\right]}}{\sqrt{\det B\left(y\right)}}$$
$$\times\int\mathcal{F}_z^{-1}\left(\left[\frac{\left\langle i\partial_z,B\left(y\right)^{-1}\partial_z\right\rangle}{2\lambda}\right]^{M+1}f\right)\left(\zeta\right)R_{M+1}\left(-i\frac{\zeta^{\mathrm{tr}}B\left(y\right)^{-1}\zeta}{2\lambda}\right)d\zeta,$$

where

$$f\left(z,y,\lambda\right)\equiv\frac{a_\lambda\left(\Psi_y^{-1}\left(z\right),y\right)}{\det\left[\left(\partial_x\Psi\right)\left(\Psi_y^{-1}\left(z\right)\right)\right]},$$

and $B\left(y\right)=\partial_x^2\phi\left(X\left(y\right),y\right)$, *and* $X\left(y\right)$ *is the unique stationary point of* $\phi\left(\cdot_x,y\right)$ *in the support of* a, *as given in the Morse Lemma, and finally,*

$$R_{M+1}\left(ib\right)=\int_0^1 e^{itb}\left(ib\right)^{M+1}\frac{\left(1-t\right)^{M+1}}{\left(M+1\right)!}dt,\qquad \text{for } b\in\mathbb{R}.$$

The remainder term satisfies the estimate,

$$\sup_{y\in V}\left|\mathfrak{R}^{(M+1)}_{a_\lambda,\phi}(y,\lambda)\right|\leq C_M\lambda^{-\frac{n}{2}-(M+1)}\sum_{|\alpha|\leq\rho+2(M+1)}\left\|\partial_x^\alpha a_\lambda\right\|_{L^2(\mathbb{R}^n_x)\times L^\infty\left(\mathbb{R}^{d+1}_{y,\lambda}\right)},\tag{3.12}$$

where $\rho=\left\lceil\frac{n}{2}\right\rceil$ *is the smallest integer greater than* $\frac{n}{2}$, *and if* $N>M+1+\frac{n}{2}$, *then we also have the alternate bound,*

$$\sup_{y\in V}\left|\mathfrak{R}^{(M+1)}_{a_\lambda,\phi}(y,\lambda)\right|\leq C_M\lambda^{-\frac{n}{2}-M-1}\left\|\left(\mathrm{Id}-\triangle_x\right)^N a_\lambda\right\|_{L^1(\mathbb{R}^n_x)\times L^\infty\left(\mathbb{R}^n_y\right)}.\tag{3.13}$$

Proof. Take V, W, X and Ψ as in the Morse Lemma, so that

$$\phi\left(x,y\right)=\phi\left(X\left(y\right),y\right)+\frac{1}{2}\Psi\left(x,y\right)^{\mathrm{tr}}\ \left[\partial_x^2\phi\left(X\left(y\right),y\right)\right]\ \Psi\left(x,y\right),\qquad y\in V.$$

Using Lemma 23 together with a partition of unity shows that we may assume $a_\lambda\left(x,y\right)$ is supported in W for all $y\in V$. Thus a change of variables

$$z=\Psi\left(x,y\right)=\Psi_y\left(x\right),$$

gives,

$$\begin{aligned}\mathcal{I}_{a_\lambda,\phi}\left(y,\lambda\right)&=\int_{\mathbb{R}^n}e^{i\lambda\phi(x,y)}a_\lambda\left(x,y\right)dx=\int_{\mathbb{R}^n}e^{i\lambda\phi\left(\Psi_y^{-1}z,y\right)}\frac{a_\lambda\left(\Psi_y^{-1}z,y\right)}{\det\left[\left(\partial_x\Psi\right)\left(\Psi_y^{-1}\left(z\right),y\right)\right]}dz\\&=\int_{\mathbb{R}^n}e^{i\lambda\left[\phi(x_0,y_0)+\Psi\left(\Psi_y^{-1}z,y\right)^{\mathrm{tr}}\frac{\partial_x^2\phi(X(y),y)}{2}\Psi\left(\Psi_y^{-1}z,y\right)\right]}\frac{a_\lambda\left(\Psi_y^{-1}\left(z\right),y\right)}{\det\left[\left(\partial_x\Psi\right)\left(\Psi_y^{-1}\left(z\right),y\right)\right]}dz\\&=\int_{\mathbb{R}^n}e^{i\lambda\left[\phi(x_0,y_0)+z^{\mathrm{tr}}\frac{\partial_x^2\phi(X(y),y)}{2}z\right]}\frac{a_\lambda\left(\Psi_y^{-1}\left(z\right),y\right)}{\det\left[\left(\partial_x\Psi\right)\left(\Psi_y^{-1}\left(z\right),y\right)\right]}dz\\&=e^{i\lambda\phi(x_0,y_0)}\int_{\mathbb{R}^n}e^{i\lambda z^{\mathrm{tr}}\frac{\partial_x^2\phi(X(y),y)}{2}z}f\left(z,y,\lambda\right)dz,\end{aligned}$$

where

$$f(z,y,\lambda)\equiv\frac{a_\lambda\left(\Psi_y^{-1}(z),y\right)}{\det\left[(\partial_x\Psi)\left(\Psi_y^{-1}(z)\right)\right]}.$$

Now write

$$B(y)=\left(\partial_x^2\phi\right)(X(y),y), \tag{3.14}$$

and apply the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ in the variable z and its dual variable ζ to obtain

$$\mathcal{I}_{a_\lambda,\phi}(y,\lambda)=e^{i\lambda\phi(x_0,y_0)}\int_{\mathbb{R}^n}\mathcal{F}_z\left(e^{i\lambda z^{\mathrm{tr}}\frac{B(y)}{2}z}\right)(\zeta)\ \mathcal{F}_z^{-1}(f(z,y))(\zeta)\,d\zeta.$$

Using Lemma 25 with $A=\frac{\lambda}{2}B(y)$, we have,

$$\begin{aligned}\mathcal{I}_{a_\lambda,\phi}(y,\lambda)&=e^{i\lambda\phi(x_0,y_0)}\frac{\pi^{\frac{n}{2}}e^{i\,\mathrm{sgn}\,B(y)\frac{\pi}{4}}}{\sqrt{\det\frac{\lambda}{2}B(y)}}\int_{\mathbb{R}^n}e^{-i\frac{\zeta^{\mathrm{tr}}B(y)^{-1}\zeta}{2\lambda}}\mathcal{F}_z^{-1}(f(z,y))(\zeta)\,d\zeta\\&=\left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}}\frac{e^{i\,\mathrm{sgn}\,B(y)\frac{\pi}{4}}e^{i\lambda\phi(x_0,y_0)}}{\sqrt{\det B(y)}}\int_{\mathbb{R}^n}e^{-i\frac{\zeta^{\mathrm{tr}}B(y)^{-1}\zeta}{2\lambda}}\mathcal{F}_z^{-1}(f(z,y))(\zeta)\,d\zeta.\end{aligned}$$

Next we use Taylor's formula with integral remainder to obtain that for any $M>0$,

$$e^{ib}=\sum_{\ell=0}^{M}\frac{(ib)^\ell}{\ell!}+R_{M+1}(ib),$$

where

$$R_{M+1}(ib)=\int_0^1 e^{itb}(ib)^{M+1}\frac{(1-t)^{M+1}}{(M+1)!}dt\text{ and }|R_{M+1}(ib)|\leq\frac{|b|^{M+1}}{(M+2)!}$$

and so with

$$b=-\frac{\zeta^{\mathrm{tr}}B(y)^{-1}\zeta}{2\lambda},$$

we have

$$\mathcal{I}_{a_\lambda,\phi}(y,\lambda)-\left(\frac{2\pi}{\lambda}\right)^{\frac{d}{2}}\frac{e^{i\left[\mathrm{sgn}\,B(y)\frac{\pi}{4}+\lambda\phi(X(y),y)\right]}}{\sqrt{\det B(y)}}\int_{\mathbb{R}^n}\sum_{\ell=0}^{M}\frac{i^\ell}{(2\lambda)^\ell\,\ell!}\mathcal{F}_z^{-1}\left(\left\langle\partial_z^{\mathrm{tr}}B(y)^{-1}\partial_z\right\rangle^\ell f\right)(\zeta)\,d\zeta \tag{3.15}$$

$$\begin{aligned}&=\left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}}\frac{e^{i\left[\mathrm{sgn}\,B(y)\frac{\pi}{4}+\lambda\phi(X(y),y)\right]}}{\sqrt{\det B(y)}}\\&\quad\times\int_{\mathbb{R}^n}\mathcal{F}_z^{-1}\left(\left[\frac{\left\langle i\partial_z^{\mathrm{tr}}B(y)^{-1}\partial_z\right\rangle}{2\lambda}\right]^{M+1}f\right)(\zeta)\,R_{M+1}\left(-i\frac{\zeta^{\mathrm{tr}}B(y)^{-1}\zeta}{2\lambda}\right)d\zeta.\end{aligned}$$

Finally, using the Fourier inversion formula $\int_{\mathbb{R}^n}\mathcal{F}^{-1}(g)(z)\,dz=g(0)$, together with the identities

$$\begin{aligned}\Psi_y(X(y))&=\Psi(X(y),y)=0,\\\Psi_y^{-1}(0)&=X(y),\\\det\partial_x\Psi(X(y),y)&=\det\mathrm{Id}_n=1,\end{aligned}$$

from part (2) of the Morse Lemma, we obtain

$$\int_{\mathbb{R}^n}\mathcal{F}_z^{-1}\left(\left\langle\partial_z^{\mathrm{tr}}B(y)^{-1}\partial_z\right\rangle^\ell f\right)(\zeta)\,d\zeta=\left\langle\partial_z^{\mathrm{tr}}B(y)^{-1}\partial_z\right\rangle^\ell f(0),\qquad 0\leq\ell\leq M.$$

Now when $\ell=0$ we have

$$f(0)=\frac{a_\lambda\left(\Psi_y^{-1}(0),y\right)}{\det\left[\partial_x\Psi\left(\Psi_y^{-1}(0),y\right)\right]}=\frac{a_\lambda(X(y),y)}{\det\left[\partial_x\Psi(X(y),y)\right]}=a_\lambda(X(y),y).$$

From the change of variable $(x,y)\to(z,w)$ where $z=\Psi(x,y)$ and $w=y$, the Jacobian matrix in block form is,

$$\frac{\partial(z,w)}{\partial(x,y)}=\begin{bmatrix}\partial_x z&\partial_y z\\\partial_x w&\partial_y w\end{bmatrix}=\begin{bmatrix}\partial_x\Psi(x,y)&\partial_y\Psi(x,y)\\0_n&\mathrm{Id}_n\end{bmatrix},$$

and so

$$\left[\begin{array}{cc}\partial_z x & \partial_w x\\ \partial_z y & \partial_w y\end{array}\right]=\frac{\partial(x,y)}{\partial(z,w)}=\left[\begin{array}{cc}\partial_x\Psi(x,y) & \partial_y\Psi(x,y)\\ 0_n & \mathrm{Id}_n\end{array}\right]^{-1}=\frac{1}{\det\partial_x\Psi(x,y)}\left[\begin{array}{cc}\mathrm{Id}_n & -\partial_y\Psi(x,y)\\ 0_n & \partial_x\Psi(x,y)\end{array}\right].$$

Thus we have by the chain rule,

$$\begin{aligned}\left(\begin{array}{c}\partial_z\\ \partial_w\end{array}\right) &= \left[\begin{array}{cc}\partial_z x & \partial_z y\\ \partial_w x & \partial_w y\end{array}\right]\left(\begin{array}{c}\partial_x\\ \partial_y\end{array}\right)=\frac{1}{\det\partial_x\Psi(x,y)}\left[\begin{array}{cc}\mathrm{Id}_n & -\partial_y\Psi(x,y)\\ 0_n & \partial_x\Psi(x,y)\end{array}\right]^{\mathrm{tr}}\left(\begin{array}{c}\partial_x\\ \partial_y\end{array}\right)\\ &= \frac{1}{\det\partial_x\Psi(x,y)}\left[\begin{array}{cc}\mathrm{Id}_n & 0_n\\ -\partial_y\Psi(x,y) & \partial_x\Psi(x,y)\end{array}\right]\left(\begin{array}{c}\partial_x\\ \partial_y\end{array}\right)\\ &= \frac{1}{\det\partial_x\Psi(x,y)}\left(\begin{array}{c}\partial_x\\ -\partial_y\Psi(x,y)\,\partial_x+\partial_x\Psi(x,y)\,\partial_y\end{array}\right),\end{aligned}$$

i.e.,

$$\partial_z=\frac{1}{\det\partial_x\Psi(x,y)}\partial_x. \tag{3.16}$$

Thus when $\ell=1$ we have

$$\begin{aligned}\left\langle\partial_z^{\mathrm{tr}}B(y)^{-1}\partial_z\right\rangle f(0) &= \left(\partial_z^{\mathrm{tr}}B(y)^{-1}\partial_z\frac{a_\lambda\left(\Psi_y^{-1}(z),y\right)}{\det\left[\left(\partial_x\Psi\right)\left(\Psi_y^{-1}(z),y\right)\right]}\right)(0)\\ &= \left(\left\{\left[\partial_x\frac{1}{\det\partial_x\Psi(x,y)}\right]^{\mathrm{tr}}B(y)^{-1}\frac{1}{\det\partial_x\Psi(x,y)}\partial_x\right\}\frac{a_\lambda(x,y)}{\det\left[\partial_x\Psi(x,y)\right]}\right)|_{x=X(y)}\\ &= L(y,\partial_x)\frac{a_\lambda(x,y)}{\det\left[\partial_x\Psi(x,y)\right]}|_{x=X(y)},\end{aligned}$$

where

$$L(y,\partial_x)\equiv\left[\partial_x\frac{1}{\det\partial_x\Psi(x,y)}\right]^{\mathrm{tr}}B(y)^{-1}\frac{1}{\det\partial_x\Psi(x,y)}\partial_x$$

is a second order differential operator in x with coefficients depending on both x and y. More generally, the same calculation shows that for $0\leq\ell\leq M$, we have,

$$\begin{aligned}\left\langle\partial_z,B(y)^{-1}\partial_z\right\rangle^{\ell}f(0) &= \left(\left\{\left[\partial_x\frac{1}{\det\partial_x\Psi(x,y)}\right]^{\mathrm{tr}}B(y)^{-1}\frac{1}{\det\partial_x\Psi(x,y)}\partial_x\right\}^{\ell}\frac{a_\lambda(x,y)}{\det\left[\partial_x\Psi(x,y)\right]}\right)|_{x=X(y)}\\ &= L(y,\partial_x)^{\ell}\frac{a_\lambda(x,y)}{\det\left[\partial_x\Psi(x,y)\right]}|_{x=X(y)}\ .\end{aligned}$$

Thus the identity (3.15), together with the bound $\left|g_{M+1}\left(-i\frac{\xi^{\mathrm{tr}}B(y)^{-1}\xi}{2\lambda}\right)\right|\leq\frac{1}{(M+1)!}$, implies that,

$$\begin{aligned}\left|\mathcal{R}_{a_\lambda,\phi}^{(M+1)}(y,\lambda)\right| &\leq C_M\lambda^{-\frac{n}{2}-(M+1)}\left\|\mathcal{F}_z^{-1}\left(\left\langle\partial_z,B(y)^{-1}\partial_z\right\rangle^{M+1}f\right)R_{M+1}\right\|_{L^1\left(\mathbb{R}_\zeta^n\right)}\\ &\leq C_{M,n}\lambda^{-\frac{n}{2}-(M+1)}\sum_{|\alpha|\leq\rho+2(M+1)}\left\|\partial_x^\alpha a_\lambda\right\|_{L^2(\mathbb{R}_x^n)\times L^\infty\left(\mathbb{R}_y^n\right)}\ ,\end{aligned} \tag{3.17}$$

where in the last line we have used Cauchy-Schwarz, the derivative identities for $\mathcal{F}$, and Plancherel's theorem with the smallest integer $\rho=\left\lceil\frac{n}{2}\right\rceil$ greater than $\frac{n}{2}$. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\widehat{h}(\xi)\right|d\xi &= \int_{\mathbb{R}^n}\left|\widehat{h}(\xi)\right|\left(1+|\xi|^2\right)^{\rho}\left(1+|\xi|^2\right)^{-\rho}d\xi\\ &\leq \left(\int_{\mathbb{R}^n}\left|\left(1+|\xi|^2\right)^{\rho}\widehat{h}(\xi)\right|^2d\xi\right)^{\frac12}\left(\int_{\mathbb{R}^n}\left(1+|\xi|^2\right)^{-2\rho}d\xi\right)^{\frac12}\\ &\leq C_m\left(\int_{\mathbb{R}^n}\left|\left(\mathrm{Id}_n-\triangle_x\right)^{\rho}h(x)\right|^2dx\right)^{\frac12},\end{aligned}$$

for the function

$$\begin{aligned} h(x) &= \left\langle \partial_z, B(y)^{-1}\partial_z \right\rangle^{M+1} f \\ &= \left\{ \left[\partial_x \frac{1}{\det \partial_x \Psi(x,y)} \right] B(y)^{-1} \frac{1}{\det \partial_x \Psi(x,y)} \partial_x \right\}^{M+1} \frac{a_\lambda(x,y)}{\det[\partial_x \Psi(x,y)]}. \end{aligned}$$

To prove the alternate bound (3.13), we use the estimate $\left| R_{M+1}\left(-i\frac{\zeta^{\mathrm{tr}}B(y)^{-1}\zeta}{2\lambda}\right)\right| \lesssim \left|\frac{\zeta^{\mathrm{tr}}B(y)^{-1}\zeta}{2\lambda}\right|^{M+1}$ to obtain,

$$\begin{aligned} & \left\| \mathcal{F}_z^{-1}\left(\left\langle \partial_z, B(y)^{-1}\partial_z \right\rangle^{M+1} f \right) R_{M+1} \right\|_{L^1\left(\mathbb{R}^n_\zeta\right)} \\ \leq\ & \frac{1}{(M+1)!} \left\| \mathcal{F}_z^{-1}\left(\left\langle \partial_z, B(y)^{-1}\partial_z \right\rangle^{M+1} f \right) \right\|_{L^1\left(\mathbb{R}^n_\zeta\right)} \lesssim \left(\frac{1}{\lambda}\right)^{M+1} \int_{\mathbb{R}^n} |\zeta|^{2(M+1)} \left|(\mathcal{F}_z f)(\zeta)\right| d\zeta, \end{aligned}$$

where

$$\begin{aligned} (\mathcal{F}_z f)(\zeta) &= \left(\mathcal{F}_z \frac{a_\lambda\left(\Psi_y^{-1}(z), y\right)}{\det\left[(\partial_x\Psi)\left(\Psi_y^{-1}(z)\right)\right]} \right)(\zeta) = \mathcal{F}_z \varphi_y(\zeta), \\ \varphi_y(z) &\equiv \frac{a_\lambda\left(\Psi_y^{-1}(z), y\right)}{\det\left[(\partial_x\Psi)\left(\Psi_y^{-1}(z)\right)\right]}. \end{aligned}$$

From the estimate

$$\begin{aligned} \left|\mathcal{F}_z\varphi_y(\zeta)\right| &= \left| \int_{\mathbb{R}^n} e^{ix\cdot\zeta} \varphi_y(x)\, dx \right| = \left| \int_{\mathbb{R}^n} \left[\left(\frac{\mathrm{Id} - \triangle_x}{1+|\zeta|^2} \right)^N e^{ix\cdot\zeta} \right] \varphi_y(x)\, dx \right| \\ &= \frac{1}{\left(1+|\zeta|^2\right)^N} \left| \int_{\mathbb{R}^n} e^{ix\cdot\zeta} \left(\mathrm{Id} - \triangle_x\right)^N \varphi_y(x)\, dx \right| \leq \left\| \left(\mathrm{Id} - \triangle_x\right)^N \varphi_y \right\|_{L^1} \frac{1}{\left(1+|\zeta|^2\right)^N}, \end{aligned}$$

we have for $N > M+1+\frac{n}{2}$ that

$$\begin{aligned} & \left(\frac{1}{\lambda}\right)^{M+1} \int_{\mathbb{R}^n} |\zeta|^{2M+2} \left|(\mathcal{F}_z f)(\zeta)\right| d\zeta \lesssim \left(\frac{1}{\lambda}\right)^{M+1} \left\| \left(\mathrm{Id} - \triangle_x\right)^N \varphi_y \right\|_{L^1(\mathbb{R}^n_x)\times L^\infty\left(\mathbb{R}^n_y\right)} \int_{\mathbb{R}^n} \frac{|\zeta|^{2M+2}}{\left(1+|\zeta|^2\right)^N} d\zeta \\ & \lesssim \left(\frac{1}{\lambda}\right)^{M+1} \left\| \left(\mathrm{Id} - \triangle_x\right)^N \varphi_y \right\|_{L^1(\mathbb{R}^n_x)\times L^\infty\left(\mathbb{R}^n_y\right)} \lesssim \left(\frac{1}{\lambda}\right)^{M+1} \left\| \left(\mathrm{Id} - \triangle_x\right)^N a_\lambda \right\|_{L^1(\mathbb{R}^n_x)\times L^\infty\left(\mathbb{R}^n_y\right)}. \end{aligned}$$

We conclude that,

$$\begin{aligned} \left| \mathcal{R}^{(M+1)}_{a_\lambda,\phi}(y,\lambda) \right| &\leq C_M \lambda^{-\frac{n}{2}-(M+1)} \left\| \mathcal{F}_z^{-1}\left(\left\langle \partial_z, B(y)^{-1}\partial_z \right\rangle^{M+1} f \right) g_{M+1} \right\|_{L^1\left(\mathbb{R}^n_\zeta\right)} \\ &\leq C_M \lambda^{-\left(M+1+\frac{n}{2}\right)} \left\| \left(\mathrm{Id} - \triangle_x\right)^N a_\lambda \right\|_{L^1(\mathbb{R}^n_x)\times L^\infty\left(\mathbb{R}^n_y\right)}, \qquad \text{for } N > M+1+\frac{n}{2}. \end{aligned}$$

□

Remark 29. *The identity* $\partial_x\Psi(X(y), y) = \mathrm{Id}_n$ *implies that* $\det[\partial_x\Psi(X(y), y)] = 1$*. Thus for* $\ell = 1$ *we have*

$$\begin{aligned} & \partial_x \left\{ \frac{1}{\det \partial_x\Psi(x,y)} B(y)^{-1} \frac{1}{\det \partial_x\Psi(x,y)} \partial_x \frac{a_\lambda(x,y)}{\det[\partial_x\Psi(x,y)]} \right\} \\ =\ & B(y)^{-1} \left\{ -2\left(\det \partial_x\Psi(x,y)\right)^{-3} \partial_x \det \partial_x\Psi(x,y) + \partial_x^2 \left[\left(\det[\partial_x\Psi(x,y)]\right)^{-1} a_\lambda(x,y) \right] \right\}, \end{aligned}$$

where $\partial_x^2 \left[(\det [\partial_x \Psi(x,y)])^{-1} a_\lambda(x) \right]$ *is*

$$\begin{aligned} & 2 (\det [\partial_x \Psi(x,y)]) \left|^{-3} \partial_x \det \partial_x \Psi(x,y) \right|^2 a_\lambda(x,y) \\ & - (\det [\partial_x \Psi(x,y)])^{-2} \partial_x^2 \det \partial_x \Psi(x,y) a_\lambda(x,y) - (\det [\partial_x \Psi(x)])^{-2} \partial_x \det \partial_x \Psi(x,y) \partial_x a_\lambda(x,y) \\ & - (\det [\partial_x \Psi(x,y)])^{-2} \partial_x \det \partial_x \Psi(x,y) \partial_x a_\lambda(x,y) + (\det [\partial_x \Psi(x,y)])^{-1} \partial_x^2 a_\lambda(x,y), \end{aligned}$$

and so when we evaluate at $x = X(y)$*, we obtain that* $(\det [\partial_x \Psi(x,y)])^{-1} \partial_x^2 a(x,y)$ *equals* $\partial_x^2 a(X(y), y)$*, and hence,*

$$\mathcal{P}_{a,\phi}^{(1)}(y,\lambda) = \frac{i^\ell}{(2\lambda)^\ell \ell!} \left(\frac{2\pi}{\lambda} \right)^{\frac{n}{2}} \frac{e^{i\left[\operatorname{sgn} B(y) \frac{\pi}{4} + \lambda \phi(X(y),y)\right]}}{\sqrt{\det B(y)}} \left\{ \partial_x^2 a(X(y), y) + O\left(\|\partial_x a_\lambda\|_{L^\infty(\mathbb{R}_x^n)} + \|a_\lambda\|_{L^\infty(\mathbb{R}_x^n)} \right) \right\}.$$

Thus every gain of $\frac{1}{\lambda}$ *costs* ***two*** *derivatives of* a_λ *in* x *(ignoring the contribution from* $\|\partial_x a_\lambda\|_{L^\infty(\mathbb{R}_x^n)} + \|a_\lambda\|_{L^\infty(\mathbb{R}_x^n)}$*), which dictates our definition of the parameter* d *in the subform (4.3) below.*

Note that we can write the formula for $\mathfrak{P}_{a_\lambda,\phi}^{(\ell)}(y,\lambda)$ compactly as

$$\mathfrak{P}_{a_\lambda,\phi}^{(\ell)}(y,\lambda) = \left(\frac{2\pi}{\lambda} \right)^{\frac{n}{2}} \frac{i^\ell}{(2\lambda)^\ell \ell!} \frac{e^{i\left[\operatorname{sgn} B(y) \frac{\pi}{4} + \lambda \phi(X(y),y)\right]}}{\sqrt{\det B}} \left(\left\{ L^{-1} \partial_x B L^{-1} \partial_x \right\}^\ell \frac{a(x,y)}{\det L} \right) |_{x=X(y)}, \tag{3.18}$$

where

$$L \equiv \partial_x \Psi(x,y) \text{ and } B \equiv B(y) = \left(\partial_x^2 \phi \right)(X(y), y). \tag{3.19}$$

4. Starting the proof of the probabilistic extension conjecture

We must prove the truncated probabilistic extension inequality (1.8),

$$\mathbb{E}_{2^{\mathcal{G}}}^\mu \left\| T \sum_{I \in \mathcal{G}[U]} a_I \triangle_{I;\kappa}^\eta f \right\|_{L^p(\lambda_n)} \leq C \|f\|_{L^p\left(B\left(0,\frac{1}{2}\right)\right)}, \qquad p > \frac{2n}{n-1}.$$

However, we will instead begin by setting out to prove the much stronger truncated *deterministic* extension inequality (1.9),

$$\left\| T \sum_{I \in \mathcal{G}[U]} \triangle_{I;\kappa}^\eta f \right\|_{L^p(\lambda_n)} \leq C \|f\|_{L^p\left(B\left(0,\frac{1}{2}\right)\right)},$$

and only when we run into difficulty proving this, will we revert to using expectation. Thus we begin by considering its equivalent bilinear inequality

$$\left| \left\langle T \sum_{I \in \mathcal{G}[U]} \triangle_{I;\kappa}^\eta f, g \right\rangle \right| \lesssim \|f\|_{L^p} \|g\|_{L^{p'}} .$$

Our initial splitting of the above bilinear form is modeled after that in two weight testing theory using (1.22),

$$\begin{aligned} \left\langle T \sum_{I \in \mathcal{G}[U]} \triangle_{I;\kappa}^\eta f, g \right\rangle &= \sum_{(I,J) \in \mathcal{G}[U] \times \mathcal{D}} \left\langle T \triangle_{I;\kappa}^{n-1,\eta} f, \triangle_{J;\kappa}^{n,\eta} g \right\rangle \\ &= \left\{ \sum_{(I,J) \in \mathcal{P}_0} + \sum_{(I,J) \in \mathcal{R}} + \sum_{m=1}^{cs} \sum_{(I,J) \in \mathcal{P}_m} + \sum_{(I,J) \in \mathcal{X}} \right\} \left\langle T \triangle_{I;\kappa}^{n-1,\eta} f, \triangle_{J;\kappa}^{n,\eta} g \right\rangle \\ &\equiv \mathsf{B}_{\text{below}}(f,g) + \mathsf{B}_{\text{above}}(f,g) + \mathsf{B}_{\text{disjoint}}(f,g) + \mathsf{B}_{\text{distal}}(f,g). \end{aligned}$$

We further decomposed the pairs $\mathcal{P}_0$ and $\mathcal{P}_m$ in (3.3) and (3.4) according to the oscillation properties of the inner product

$$\left\langle T \triangle_{I;\kappa}^{n-1,\eta} f, \triangle_{J;\kappa}^{n,\eta} g \right\rangle = \left\langle T h_{I;\kappa}^{n-1,\eta}, h_{J;\kappa}^{n,\eta} \right\rangle \left\langle (S_{\kappa,\eta})^{-1} f, h_{I;\kappa}^{n-1} \right\rangle \left\langle (S_{\kappa,\eta})^{-1} g, h_{J;\kappa}^n \right\rangle,$$

and we will continue to do this for $\mathcal{R}$ and $\mathcal{X}$ when needed below.

(1) The below form $\mathsf{B}_{\text{below}}(f,g)$ combines stationary phase with either integration by parts or moment vanishing, and only its subform $\mathsf{B}_{\text{below}}^{k,d}(f,g)$ for $k,d\geq 0$ requires the strict inequality $p>\frac{2n}{n-1}$. Moreover, the subforms with $d\leq 0$ can be controlled by relatively simple arguments when $p>\frac{2n}{n-1}$.
(2) The above form $\mathsf{B}_{\text{above}}(f,g)$ is less critical and easier to handle in that it doesn't use stationary phase, and is in fact bounded for all $1<p<\infty$.
(3) The disjoint form $\mathsf{B}_{\text{disjoint}}(f,g)$ is handled similarly in some places, and made easier in those places due to the fact that stationary phase is not needed, because the critical point of the phase lies outside the support of the amplitude. However, in those difficult places where large numbers of inner products are resonant, i.e. without either appropriate oscillation or smoothness, the use of *probability* is required in an interpolation argument between L^2 and L^4 estimates.
(4) The distal form $\mathsf{B}_{\text{distal}}(f,g)$ is handled as an extreme case of the disjoint form.

We have

$$\left|\left\langle T\,\triangle_{I;\kappa}^{n-1,\eta}f,\triangle_{J;\kappa}^{n,\eta}g\right\rangle_{\omega}\right|=\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\left|\left\langle\left(S_{\kappa,\eta}^{\sigma}\right)^{-1}f,h_{I;\kappa}^{n-1}\right\rangle\right|\left|\left\langle\left(S_{\kappa,\eta}^{\omega}\right)^{-1}g,h_{J;\kappa}^{n}\right\rangle\right| \tag{4.1}$$
$$\approx\left\{\frac{\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle_{\omega}\right|}{\sqrt{|I|\,|J|}}\right\}\left\{\int_{\mathbb{R}^{n-1}}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|d\sigma(x)\right\}\left\{\int_{\mathbb{R}^{n}}\left|\triangle_{J;\kappa}^{n,\eta}g(\xi)\right|d\omega(\xi)\right\},$$

since

$$\int_{\mathbb{R}^{n-1}}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|d\sigma(x)=\int_{\mathbb{R}^{n-1}}\left|\left\langle\left(S_{\kappa,\eta}\right)^{-1}f,h_{I;\kappa}^{n-1}\right\rangle h_{I;\kappa}^{n-1,\eta}\right|d\sigma(x) \tag{4.2}$$
$$\approx\left|\left\langle\left(S_{\kappa,\eta}^{\sigma}\right)^{-1}f,h_{I;\kappa}^{n-1}\right\rangle\right|\left\|h_{I;\kappa}^{n-1,\eta}\right\|_{L^1(\sigma)}\approx\left|\left\langle\left(S_{\kappa,\eta}\right)^{-1}f,h_{I;\kappa}^{n-1}\right\rangle\right|\sqrt{|I|},$$
$$\int_{\mathbb{R}^{n}}\left|\triangle_{J;\kappa}^{n,\eta}g(\xi)\right|d\omega(\xi)\approx\left|\left\langle\left(S_{\kappa,\eta}\right)^{-1}g,h_{J;\kappa}^{n}\right\rangle\right|\sqrt{|J|}.$$

Thus we now turn to estimating the inner product

$$\left\langle Th_{I;\kappa}^{n-1},h_{J;\kappa}^{n}\right\rangle=\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^{n-1}}e^{i\Phi(x)\cdot\xi}h_{I;\kappa}^{n-1,\eta}(x)\,dx\right\}h_{J;\kappa}^{n,\eta}(\xi)\,d\xi,$$

and then using these inner product estimates, we will bound the two bilinear forms $\mathsf{B}_{\text{below}}(f,g)$ and $\mathsf{B}_{\text{above}}(f,g)$, along with some of the subforms of $\mathsf{B}_{\text{disjoint}}(f,g)$ and $\mathsf{B}_{\text{distal}}(f,g)$, namely those comprising the upper disjoint and distal forms $\mathsf{B}_{\text{disjoint}}^{\text{upper}}(f,g)$ and $\mathsf{B}_{\text{distal}}^{\text{upper}}(f,g)$ (defined later).

In fact, if we denote by $|\mathsf{B}_{\text{below}}|(f,g)$, $|\mathsf{B}_{\text{above}}|(f,g)$, $\left|\mathsf{B}_{\text{disjoint}}^{\text{upper}}\right|(f,g)$ and $|\mathsf{B}_{\text{distal}}^{\text{upper}}|(f,g)$ the forms $\mathsf{B}_{\text{below}}(f,g)$, $\mathsf{B}_{\text{above}}(f,g)$, $\mathsf{B}_{\text{disjoint}}^{\text{upper}}(f,g)$ and $\mathsf{B}_{\text{distal}}^{\text{upper}}(f,g)$ with absolute values taken inside the sum of inner products, then we will prove the following 'deterministic' estimate in which probability plays no role.

Proposition 30. *For $p>\frac{2n}{n-1}$ we have*

$$|\mathsf{B}_{\text{below}}|(f,g)+|\mathsf{B}_{\text{above}}|(f,g)+\left|\mathsf{B}_{\text{disjoint}}^{\text{upper}}\right|(f,g)+|\mathsf{B}_{\text{distal}}^{\text{upper}}|(f,g)\lesssim\|f\|_{L^p(\mathbb{R}^{n-1})}\|g\|_{L^{p'}(\mathbb{R}^n)}\ .$$

Proof. This follows immediately from (6.1), (7.1), (8.4) and (8.5) below. □

Note that the small positive constant η in the construction of the smooth Alpert wavelets is fixed throughout the estimates below, and so powers of $\frac{1}{\eta}$ depending on n and κ will often be absorbed into the notation of approximate inequality $\lesssim$.

Notation 31. *In an inner product of the form $\langle T\varphi,\psi\rangle$, we refer to φ as the* amplitude *function, and to ψ as the* pairing *function.*

4.1. **Pigeonholing into bilinear subforms.** Recall the decomposition (with bounded overlap) of the pairs $(I,J)\in\mathcal{G}[U]\times\mathcal{D}$ of dyadic cubes introduced in (1.22),

$$\mathcal{G}[U]\times\mathcal{D}=\mathcal{P}_0\ \cup\ \bigcup_{m=0}^{\infty}\mathcal{P}_m\ \cup\ \mathcal{R}\ \cup\ \mathcal{X}\ ,$$

where

$$\begin{aligned}\mathcal{P}_0 &\equiv \{(I,J)\in\mathcal{G}[U]\times\mathcal{D}:\pi_{\tan}(J)\subset\Phi(C_{\text{pseudo}}I)\}\ ,\\ \mathcal{P}_m &\equiv \left\{(I,J)\in\mathcal{G}[U]\times\mathcal{D}:2^{m+1}I\subset U\text{ and }\pi_{\tan}(J)\subset\Phi\left(2^{m+1}C_{\text{pseudo}}I\right)\setminus\Phi\left(2^mC_{\text{pseudo}}I\right)\right\},\qquad m\in\mathbb{N}\ ,\\ \mathcal{R} &\equiv \{(I,J)\in\mathcal{G}[U]\times\mathcal{D}:\Phi(I)\subset\pi_{\tan}(C_{\text{pseudo}}J)\}\ .\end{aligned}$$

In treating the below form $\mathsf{B}_{\text{below}}(f,g)$, we will consider the inner products

$$\begin{aligned}\left\langle T_\sigma\bigtriangleup_{I;\kappa}^{n-1,\eta}f,\bigtriangleup_{J;\kappa}^{n,\eta}g\right\rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^{n-1}}\bigtriangleup_{I;\kappa}^{n-1,\eta}f(x)\,e^{-i\Phi(x)\cdot\xi}dx\bigtriangleup_{J;\kappa}^{n,\eta}g(\xi)\,d\xi=\left\langle T_\sigma h_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{\eta,\omega}\right\rangle\left\langle f,h_{I;\kappa}^{n-1,\eta}\right\rangle\left\langle g,h_{J;\kappa}^{n,\eta}\right\rangle\ ,\\ \left\langle T_\sigma h_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^{n-1}}h_{I;\kappa}^{n-1,\eta}(x)\,e^{-i\Phi(x)\cdot\xi}dx\,h_{J;\kappa}^{n,\eta}(\xi)\,d\xi,\end{aligned}$$

for $(I,J)\in\mathcal{P}_0\subset\mathcal{G}[U]\times\mathcal{D}$, and as in (3.3), we further decompose the index set $\mathcal{P}_0$ of pairs by pigeonholing the side length of J and its distance from the origin relative to $\frac{1}{\ell(I)^2}$, the reciprocal of the 'depth' of the spherical 'cap' $\Phi(I)$:

$$\begin{aligned}&\mathcal{P}_0=\bigcup_{k\in\mathbb{Z}}\bigcup_{d\in\mathbb{Z}}^{\infty}\mathcal{P}_0^{k,d},\text{ where}\\ &\mathcal{P}_0^{k,d}\equiv\left\{(I,J)\in\mathcal{P}_0:\ell(J)=2^k,\text{ and }2^d\leq\ell(I)^2\operatorname{dist}(0,J)\leq2^{d+1}\right\},\\ &\qquad\text{for }k,d\in\mathbb{Z}.\end{aligned}$$

Then we define the associated subforms,

$$\mathsf{B}_{\text{below}}^{k,d}(f,g)\equiv\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left\langle T_S h_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle. \tag{4.3}$$

We decompose the disjoint form $\mathsf{B}_{\text{disjoint}}(f,g)$ into subforms $\mathsf{B}_{\text{disjoint}}^{k,d,m}(f,g)$ similar to that done for the below form $\mathsf{B}_{\text{below}}(f,g)$. Recall that in (3.4), for each $m\geq0$, we decomposed the index set

$$\mathcal{P}_m\equiv\left\{(I,J)\in\mathcal{G}[U]\times\mathcal{D}:2^{m+1}I\subset U\text{ and }\pi_{\tan}(J)\subset\Phi\left(2^{m+1}C_{\text{pseudo}}I\right)\setminus\Phi\left(2^mC_{\text{pseudo}}I\right)\right\},\quad 1\leq m\leq cs\ ,$$

of pairs by pigeonholing the side length of J and its distance from the origin relative to $\frac{1}{\ell(I)^2}$, the reciprocal of the 'depth' of the spherical set $\Phi(I)$:

$$\begin{aligned}&\mathcal{P}_m=\bigcup_{k\in\mathbb{Z}}\bigcup_{d\in\mathbb{Z}}^{\infty}\mathcal{P}_m^{k,d},\text{ where}\\ &\mathcal{P}_m^{k,d}\equiv\left\{(I,J)\in\mathcal{P}_m:\ell(J)=2^k,\text{ and }2^d\leq\ell(I)^2\operatorname{dist}(0,J)\leq2^{d+1}\right\},\\ &\qquad\text{for }k,d\in\mathbb{Z},\end{aligned}$$

and now we define the disjoint subforms,

$$\mathsf{B}_{\text{disjoint}}^{k,d,m}(f,g)\equiv\sum_{(I,J)\in\mathcal{P}_m^{k,d}}\left\langle T\bigtriangleup_{I;\kappa}^{n-1,\eta}f,\bigtriangleup_{J;\kappa}^{n,\eta}g\right\rangle. \tag{4.4}$$

We point out that in those inner products in the disjoint form with resonance, such as when $k=0$ and $m=-d$, we need analogues for smooth Alpert wavelets of the traditional L^2 and L^4 estimates averaged over involutive smooth Alpert multipliers. We then write

$$\mathsf{B}_{\text{disjoint}}^{\text{upper}}(f,g)\equiv\sum_{k\in\mathbb{Z}}\sum_{d\geq0}\sum_{m\in\mathbb{N}}\mathsf{B}_{\text{disjoint}}^{k,d,m}(f,g)\ \text{ and }\mathsf{B}_{\text{disjoint}}^{\text{lower}}(f,g)\equiv\sum_{k\in\mathbb{Z}}\sum_{d<0}\sum_{m\in\mathbb{N}}\mathsf{B}_{\text{disjoint}}^{k,d,m}(f,g)\,.$$

We defer the analogous pigeonholed decompositions for the above form $\mathsf{B}_{\text{above}}(f,g)$ and the distal form $\mathsf{B}_{\text{distal}}(f,g)$ until needed. Now we turn to the four *principles of decay* used on the smooth Alpert inner products $\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle$, followed in the next subsection with the interpolation estimates.

4.2. **Decay principles.** We introduce four different principles of decay in the oscillatory kernel of the Fourier transform, namely

(1) radial integration by parts,
(2) moment vanishing of smooth Alpert wavelets (for both $h_{I;\kappa}^{n-1,\eta}$ and $h_{J;\kappa}^{n,\eta}$),
(3) stationary phase of oscillatory integrals,
(4) and tangential integration by parts.

These four principles of decay will be used as building blocks for compound principles of decay, which are obtained by iterating the exact formulas for each principle, *before* taking absolute values inside the resulting integrals, in order to obtain estimates. These estimates are then used with square function techniques as in [SaWi] to bound the three forms $\mathsf{B}_{\mathrm{below}}(f,g)$, $\mathsf{B}_{\mathrm{disjoint}}(f,g)$ and $\mathsf{B}_{\mathrm{above}}(f,g)$. However, in order to handle *resonant* subforms of $\mathsf{B}_{\mathrm{disjoint}}(f,g)$, we require an additional decay principle involving interpolation of L^2 and L^4 estimates for smooth Alpert pseudoprojections, that is described in the next subsection.

Our baseline is the following rather trivial L^1 estimate, which we refer to as the *crude* estimate,

$$\begin{aligned}(4.5)\qquad \left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right| &\leq \left\|h_{I;\kappa}^{n-1,\eta}\right\|_{L^1(\sigma)}\left\|h_{J;\kappa}^{n,\eta}\right\|_{L^1}\approx\sqrt{|I|\,|J|}\ ,\\ \left|\left\langle T\,\triangle_{I;\kappa}^{n-1,\eta}f,\triangle_{J;\kappa}^{n,\eta}g\right\rangle_{\omega}\right| &\leq \left\|\triangle_{I;\kappa}^{n-1,\eta}f\right\|_{L^1}\left\|\triangle_{J;\kappa}^{n,\eta}g\right\|_{L^1}\approx\sqrt{|I|\,|J|}\left|\left\langle f,h_{I;\kappa}^{n-1,\eta}\right\rangle\left\langle g,h_{J;\kappa}^{n,\eta}\right\rangle\right|\ ,\end{aligned}$$

where we have used (4.2) at the end of the second line.

4.2.1. *Radial integration by parts.* First we improve upon the crude estimate (4.5) when $(I,J)\in P_0^{k,0}$ with $k>0$, i.e. $\ell(J)=2^k$, namely we show that

$$\begin{aligned}(4.6)\qquad \left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right| &\leq C_N 2^{-kN}\left\|h_{I;\kappa}^{n-1,\eta}\right\|_{L^1}\left\|h_{J;\kappa}^{n,\eta}\right\|_{L^1}\approx 2^{-kN}\sqrt{|I|\,|J|}\ ,\\ \left|\left\langle T\,\triangle_{I;\kappa}^{n-1,\eta}f,\triangle_{J;\kappa}^{n,\eta}g\right\rangle\right| &\leq C_N 2^{-kN}\left\|\triangle_{I;\kappa}^{n-1,\eta}f\right\|_{L^1}\left\|\triangle_{J;\kappa}^{n,\eta}g\right\|_{L^1}\approx 2^{-kN}\sqrt{|I|\,|J|}\left|\left\langle f,h_{I;\kappa}^{n-1,\eta}\right\rangle\left\langle g,h_{J;\kappa}^{n,\eta}\right\rangle\right|\ .\end{aligned}$$

To see this, recall the change of variables (3.6) made earlier,

$$\begin{aligned}\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^{n-1}}e^{i\Phi(x)\cdot\xi}h_{I;\kappa}^{n-1,\eta}(x)\,h_{J;\kappa}^{n,\eta}(\xi)\,dxd\xi\\ = &\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}e^{i\lambda\phi(x,y)}\varphi_I^{\eta}(x)\,\widetilde{\psi}_J^{\eta}(y,\lambda)\,dxdyd\lambda,\end{aligned}$$

where

$$\begin{aligned}\phi(x,y) &\equiv \Phi(x)\cdot\Phi(y)\,,\\ \varphi_I^{\eta}(x) &\equiv h_{I;\kappa}^{n-1,\eta}(x)\ \text{ and }\ \psi_J^{\eta}(\xi)=h_{J;\kappa}^{n,\eta}(\xi)\,,\\ \widetilde{\psi}_J^{\eta}(y,\lambda) &\equiv h_{J;\kappa}^{n,\eta}\left(\lambda y,\lambda\sqrt{1-|y|^2}\right)\frac{\lambda^{n-1}}{\sqrt{1-|y|^2}}.\end{aligned}$$

We use the formula

$$\left(\frac{1}{\phi(x,y)}\partial_\lambda\right)^N e^{i\lambda\phi(x,y)}=e^{i\lambda\phi(x,y)},$$

to obtain the equality,

$$(4.7)\qquad \left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle=\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{e^{i\lambda\phi(x,y)}}{\phi(x,y)^N}\varphi_I^{\eta}(x)\,\partial_\lambda^N\widetilde{\psi}_J^{\eta}(y,\lambda)\,dxdyd\lambda,$$

which can then be estimated by

$$\begin{aligned}(4.8)\qquad \left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right| &\lesssim \|\varphi_I^{\eta}\|_{L^1}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\left|\partial_\lambda^N\widetilde{\psi}_J^{\eta}(y,\lambda)\right|dyd\lambda\\ \lesssim\ &\|\varphi_I^{\eta}\|_{L^1}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\left|\partial_\xi^N\widetilde{\psi}_J^{\eta}(y,\lambda)\right|\left(\min\left\{\frac{1}{\eta\ell(J)},\frac{1}{\lambda}\right\}\right)^N dyd\lambda\\ \approx\ &\left(\frac{1}{\eta\ell(J)}\right)^N\|\varphi_I^{\eta}\|_{L^1}\left\|\partial_\xi^N\widetilde{\psi}_J^{\eta}\right\|_{L^1}\approx 2^{-kN}\|\varphi_I^{\eta}\|_{L^1}\left\|\partial_\xi^N\widetilde{\psi}_J^{\eta}\right\|_{L^1}\approx 2^{-kN}\sqrt{|I|\,|J|},\end{aligned}$$

which gives both lines in (4.6).

4.2.2. *Vanishing moments of smooth Alpert wavelets.* Now we improve upon the crude estimate (4.5) when $(I,J)\in P_0^{k,0}$ with $k<0$, i.e. $\ell(J)=2^k$, namely we show that

$$\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\leq C_\kappa 2^{-|k|\kappa}\left\|h_{I;\kappa}^{n-1,\eta}\right\|_{L^1}\left\|h_{J;\kappa}^{n,\eta}\right\|_{L^1}\approx 2^{-|k|\kappa}\sqrt{|I|\,|J|}\ , \tag{4.9}$$
$$\left|\left\langle T\,\triangle_{I;\kappa}^{n-1,\eta}f,\triangle_{J;\kappa}^{n,\eta}g\right\rangle_\omega\right|\leq C_\kappa 2^{-|k|\kappa}\left\|\triangle_{I;\kappa}^{n-1,\eta}f\right\|_{L^1}\left\|\triangle_{J;\kappa}^{n,\eta}g\right\|_{L^1}\approx 2^{-|k|\kappa}\sqrt{|I|\,|J|}\left|\left\langle f,h_{I;\kappa}^{n-1,\eta}\right\rangle\left\langle g,h_{J;\kappa}^{n,\eta}\right\rangle\right|\ .$$

For any entire function f, Taylor's formula with integral remainder applied to $t\to f(tz)$ gives,

$$\begin{aligned} f(z) &= \sum_{\ell=0}^{\kappa-1}\frac{1}{\ell!}\frac{d^\ell}{dt^\ell}f(tz)\,|_{t=0}+\int_0^1\left(\frac{d^\kappa}{dt^\kappa}f(tz)\right)\frac{(1-t)^\kappa}{\kappa!}dt\\ &= \sum_{\ell=0}^{\kappa-1}\frac{1}{\ell!}f^{(\ell)}(0)\,z^\ell+\int_0^1 f^{(\kappa)}(tz)\,z^\kappa\frac{(1-t)^\kappa}{\kappa!}dt, \end{aligned}$$

which shows that for any $\kappa\in\mathbb{N}$ and $b\in\mathbb{R}$, we have

$$e^{ib}=\sum_{\ell=0}^{\kappa-1}\frac{(ib)^\ell}{\ell!}+R_\kappa(ib)\,, \tag{4.10}$$

where

$$R_\kappa(ib)=\int_0^1 e^{itb}(ib)^\kappa\frac{(1-t)^\kappa}{\kappa!}dt\text{ and }|R_\kappa(ib)|\leq\frac{|b|^\kappa}{(\kappa+1)!}. \tag{4.11}$$

We also have

$$\begin{aligned} \left|\partial_b^\ell R_\kappa(ib)\right| &\lesssim \frac{|b|^{\kappa-\ell}}{(\kappa+1)!},\qquad\text{for }0\leq\ell<\kappa,\\ \partial_b^\ell R_\kappa(ib) &= \partial_b^\ell e^{ib}=i^\ell e^{ib},\qquad\text{for }\ell\geq\kappa. \end{aligned} \tag{4.12}$$

Now let c_J denote the center of the cube J and write,

$$e^{-i\Phi(x)\cdot\xi}=e^{-i\Phi(x)\cdot c_J}e^{-i\Phi(x)\cdot(\xi-c_J)}=e^{-i\Phi(x)\cdot c_J}\left\{\sum_{\ell=0}^{\kappa-1}\frac{(-i\Phi(x)\cdot(\xi-c_J))^\ell}{\ell!}+R_\kappa(-i\Phi(x)\cdot(\xi-c_J))\right\}.$$

Note that

$$e^{-i\Phi(x)\cdot c_J}R_\kappa(-i\Phi(x)\cdot(\xi-c_J))=\int_0^1 e^{-i\Phi(x)\cdot c_J}e^{-it\Phi(x)\cdot(\xi-c_J)}(-i\Phi(x)\cdot(\xi-c_J))^\kappa\frac{(1-t)^\kappa}{\kappa!}dt$$

Since $h_{J;\kappa}^{n,\eta}$ has vanishing moments up to order less than κ, we obtain

$$\begin{aligned} \left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^{n-1}}e^{-i\Phi(x)\cdot\xi}h_{I;\kappa}^{n-1,\eta}(x)\,dx\,h_{J;\kappa}^{n,\eta}(\xi)\,d\xi\\ &= \int_{\mathbb{R}^{n-1}}e^{-i\Phi(x)\cdot c_J}h_{I;\kappa}^{n-1,\eta}(x)\left\{\int_{\mathbb{R}^n}\left[\sum_{\ell=0}^{\kappa-1}\frac{(-i\Phi(x)\cdot(\xi-c_J))^\ell}{\ell!}+R_\kappa(-i\Phi(x)\cdot(\xi-c_J))\right]h_{J;\kappa}^{n,\eta}(\xi)\,d\xi\right\}dx\\ &= \int_{\mathbb{R}^{n-1}}e^{-i\Phi(x)\cdot c_J}h_{I;\kappa}^{n-1,\eta}(x)\left\{\int_{\mathbb{R}^n}R_\kappa(-i\Phi(x)\cdot(\xi-c_J))\,h_{J;\kappa}^{n,\eta}(\xi)\,d\xi\right\}dx. \end{aligned} \tag{4.13}$$

From the bound for R_κ in (4.11) with $b=-\Phi(x)\cdot(\xi-c_J)$, we have

$$\begin{aligned} \left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right| &\leq \int\left|h_{I;\kappa}^{n-1,\eta}(x)\right|\int_{\mathbb{R}^n}\frac{|\Phi(x)\cdot(\xi-c_J)|^\kappa}{(\kappa+1)!}\left|h_{J;\kappa}^{n,\eta}(\xi)\right|d\xi dx\\ &\lesssim \ell(J)^\kappa\left\|\varphi_I^\eta\right\|_{L^1}\left\|\psi_J^\eta\right\|_{L^1}\approx 2^{-|k|\kappa}\sqrt{|I|\,|J|}. \end{aligned} \tag{4.14}$$

4.2.3. *Stationary phase with bounds.* Now we improve upon the crude estimate (4.5) when $(I,J)\in P_0^{0,d}$ with $d\geq 0$, i.e. $J\subset\mathcal{K}(I)$, $\ell(J)=1$, and $\ell(I)^2\operatorname{dist}(0,J)\approx 2^d$, namely we show,

$$\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim 2^{-d\frac{n-1}{2}}\left(1+2^{-d}\left(\frac{1}{\ell(I)^2}\right)^{\tau}\right)\sqrt{|I|\,|J|}\ , \tag{4.15}$$

$$\left|\left\langle T\,\triangle_{I;\kappa}^{n-1,\eta}f,\triangle_{J;\kappa}^{n,\eta}g\right\rangle_{\omega}\right|\lesssim 2^{-d\frac{n-1}{2}}\left(1+2^{-d}\left(\frac{1}{\ell(I)^2}\right)^{\tau}\right)\sqrt{|I|\,|J|}\left|\left\langle f,h_{I;\kappa}^{n-1,\eta}\right\rangle\left\langle g,h_{J;\kappa}^{n,\eta}\right\rangle\right|\ ,$$

where $0<\tau\leq 1$. For this, recall the change of variables in (3.6) and (3.7),

$$\begin{aligned}\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^{n-1}}e^{i\Phi(x)\cdot\xi}h_{I;\kappa}^{n-1,\eta}(x)\,h_{J;\kappa}^{n,\eta}(\xi)\,dxd\xi\\&=\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\left\{\int_{\mathbb{R}^{n-1}}e^{i\lambda\phi(x,y)}\varphi_I^{\eta}(x)\,dx\right\}\widetilde{\psi}_J^{\eta}(y,\lambda)\,dyd\lambda,\end{aligned}$$

where

$$\begin{aligned}\phi(x,y)&\equiv\Phi(x)\cdot\Phi(y)\,,\\\varphi_I^{\eta}(x)&\equiv h_{I;\kappa}^{n-1,\eta}(x)\ \text{ and }\psi_J^{\eta}(\xi)=h_{J;\kappa}^{n,\eta}(\xi)\,,\\\widetilde{\psi}_J^{\eta}(y,\lambda)&\equiv\psi_J^{\eta}\left(\lambda y,\lambda\sqrt{1-|y|^2}\right)\frac{\lambda^{n-1}}{\sqrt{1-|y|^2}}.\end{aligned}$$

Applying Theorem 28 with n replaced by $n-1$ and $a_\lambda(x,y)$ equal to $\varphi_I^{\eta}(x)$, shows that the oscillatory integral

$$\mathcal{I}_{\varphi_I^{\eta},\phi}(y,\lambda)\equiv\int_{\mathbb{R}^{n-1}}e^{i\lambda\phi(x,y)}\varphi_I^{\eta}(x)\,dx,$$

satisfies

$$\mathcal{I}_{\varphi_I^{\eta},\phi}(y,\lambda)=\mathfrak{P}_{\varphi_I^{\eta},\phi}(y,\lambda)+\sum_{\ell=1}^{M}\mathfrak{P}_{\varphi_I^{\eta},\phi}^{(\ell)}(y,\lambda)+\mathfrak{R}_{\varphi_I^{\eta},\phi}^{(M+1)}(y,\lambda)\,,$$

where

$$\mathfrak{P}_{\varphi_I^{\eta},\phi}(y,\lambda)=\left(\frac{2\pi}{\lambda}\right)^{\frac{n-1}{2}}\frac{e^{i\left[\operatorname{sgn}\left[\partial_x^2\phi(X(y),y)\right]\frac{\pi}{4}+\lambda\phi(X(y),y)\right]}}{\sqrt{|\det B(y)|}}\varphi_I^{\eta}(X(y))\,, \tag{4.16}$$

and for $1\leq\ell\leq M$,

$$\begin{aligned}\mathfrak{P}_{\varphi_I^{\eta},\phi}^{(\ell)}(y,\lambda)&=\frac{i^{\ell}}{(2\lambda)^{\ell}\,\ell!}\left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}}\frac{e^{i\left[\operatorname{sgn}B(y)\frac{\pi}{4}+\lambda\phi(X(y),y)\right]}}{\sqrt{\det B(y)}}\\&\quad\times\left\{\left[\partial_x\frac{1}{\det\partial_x\Psi(X(y),y)}\right]B(y)^{-1}\frac{1}{\det\partial_x\Psi(X(y),y)}\partial_x\right\}^{\ell}\frac{\varphi_I^{\eta}(X(y))}{\det\left[\partial_x\Psi(X(y),y)\right]},\end{aligned}$$

and

$$\begin{aligned}\mathfrak{R}_{\varphi_I^{\eta},\phi}^{(M+1)}(y,\lambda)&=\left(\frac{2\pi}{\lambda}\right)^{\frac{n-1}{2}}\frac{e^{i\left[\operatorname{sgn}B(y)\frac{\pi}{4}+\lambda\phi(X(y),y)\right]}}{\sqrt{|\det B(y)|}}\\&\quad\times\int\mathcal{F}_z^{-1}\left(\left[\frac{\left\langle i\partial_z,B(y)^{-1}\partial_z\right\rangle}{2\lambda}\right]^{M+1}f\right)(\zeta)\,g_{M+1}\left(-i\frac{\zeta^{\mathrm{tr}}B(y)^{-1}\zeta}{2\lambda}\right)d\zeta,\end{aligned}$$

and where $B(y)=\partial_x^2\phi(X(y),y)$, and $X(y)$ is the unique stationary point of $\phi(\cdot_x,y)$ in the support of a, as given in the Morse Lemma, and $\rho=\left\lceil\frac{n}{2}\right\rceil$ is the smallest integer greater than $\frac{n}{2}$, and finally $g_{M+1}(b)=$

$\frac{1}{M!}\int_0^b e^t (b-t)^M \, dt$ for $b\in\mathbb{C}$. Thus at this point we have the formula,

$$\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\left\{\int_{\mathbb{R}^{n-1}} e^{i\lambda\phi(x,y)}h_{I;\kappa}^{n-1,\eta}(x)\,dx\right\}\widehat{\psi}_J^{\eta}(y,\lambda)\,dyd\lambda = \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\mathcal{I}_{\varphi_I^{\eta},\phi}(y,\lambda)\,\widetilde{\psi}_J^{\eta}(y,\lambda)\,dyd\lambda \tag{4.17}$$

In the case $\phi(x,y)\equiv\Phi(x)\cdot\Phi(y)$ we have $X(y)=y$ and

$$\begin{aligned} B(y) &= \partial_x^2\Phi(x)\cdot\Phi(y)\mid_{x=y}=\partial_x^2\sqrt{1-|x|^2}\mid_{x=y}\sqrt{1-|y|^2} \\ &= \left(-\frac{1}{\sqrt{1-|x|^2}}\operatorname{Id}_{n-1}-\frac{xx^{\mathrm{tr}}}{\left(1-|x|^2\right)^{\frac{3}{2}}}\mid_{x=y}\right)\sqrt{1-|y|^2} \\ &= -\operatorname{Id}_{n-1}-\frac{yy^{\mathrm{tr}}}{1-|y|^2}, \end{aligned}$$

so that $\operatorname{sgn}B(y)=-(n-1)$ and

$$\begin{aligned} \det B(y) &= \det\begin{bmatrix} -1-\frac{y_1^2}{1-|y|^2} & -\frac{y_1y_2}{1-|y|^2} & \cdots & -\frac{y_1y_{n-1}}{1-|y|^2} \\ -\frac{y_2y_1}{1-|y|^2} & -1-\frac{y_2^2}{1-|y|^2} & & -\frac{y_2y_{n-1}}{1-|y|^2} \\ \vdots & & \ddots & \vdots \\ -\frac{y_{n-1}y_1}{1-|y|^2} & -\frac{y_{n-1}y_1}{1-|y|^2} & \cdots & -1-\frac{y_{n-1}^2}{1-|y|^2} \end{bmatrix} \\ &= \det\frac{1}{1-|y|^2}\begin{bmatrix} -1+|y|^2-y_1^2 & -\frac{y_1y_2}{1-|y|^2} & \cdots & -y_1y_{n-1} \\ -y_2y_1 & -1+|y|^2-y_2^2 & & -y_2y_{n-1} \\ \vdots & & \ddots & \vdots \\ -y_{n-1}y_1 & -y_{n-1}y_1 & \cdots & -1+|y|^2-y_{n-1}^2 \end{bmatrix}=\frac{(-1)^{n-1}}{1-|y|^2}, \end{aligned}$$

by induction on n.

In particular then, from (3.9) and the above calculation, we have $\Psi(X(y),y)=0$, $\phi(X(y),y)$ and $\partial_x\Psi(X(y),y)=\operatorname{Id}_n$ and so

$$\mathfrak{P}_{h_{I;\kappa}^{n-1,\eta},\phi}(y,\lambda)=\left(\frac{2\pi}{\lambda}\right)^{\frac{n-1}{2}}e^{i\left[-\frac{(n-1)\pi}{4}+\lambda\right]}\sqrt{1-|y|^2}\varphi_I^{\eta}(y),$$

which can be written in the variable $\xi=\left(\lambda y,\lambda\sqrt{1-|y|^2}\right)$ as

$$\mathfrak{P}_{h_{I;\kappa}^{n-1,\eta},\phi}(\xi)=\left(\frac{2\pi}{|\xi|}\right)^{\frac{n-1}{2}}\frac{\xi_n}{|\xi|}e^{i\left(|\xi|-\frac{(n-1)\pi}{4}\right)}h_{I;\kappa}^{n-1,\eta}\left(\frac{\xi'}{|\xi|}\right),\qquad \xi'=(\xi_1,...,\xi_{n-1}).$$

We compute that for $J\in\mathcal{K}(I)$ and $\ell(I)^2\operatorname{dist}(0,J)\approx 2^d$,

$$\begin{aligned} &\left|\left\langle\mathfrak{P}_{h_{I;\kappa}^{n-1,\eta},\phi},h_{J;\kappa}^{n,\eta}\right\rangle\right|=\left|\int_{\mathbb{R}^n}\mathfrak{P}_{h_{I;\kappa}^{n-1,\eta},\phi}(\xi)\,h_{J;\kappa}^{n,\eta}(\xi)\,d\xi\right|=\left|\int_{\mathbb{R}^n}\left(\frac{2\pi}{|\xi|}\right)^{\frac{n-1}{2}}\frac{\xi_n}{|\xi|}e^{i\left(|\xi|-\frac{(n-2)\pi}{4}\right)}h_{I;\kappa}^{n-1,\eta}\left(\frac{\xi'}{|\xi|}\right)h_{J;\kappa}^{n,\eta}(\xi)\,d\xi\right| \\ \lesssim\ &\int_{\mathbb{R}^n}\left(\frac{1}{\operatorname{dist}(0,J)}\right)^{\frac{n-1}{2}}\left|h_{I;\kappa}^{n-1,\eta}\left(\frac{\xi'}{|\xi|}\right)\right|\left|h_{J;\kappa}^{n,\eta}(\xi)\right|d\xi\approx\left(\frac{1}{\operatorname{dist}(0,J)}\right)^{\frac{n-1}{2}}\int_{\mathbb{R}^n}\frac{1}{\sqrt{|I|}}\mathbf{1}_I\left(\frac{\xi'}{|\xi|}\right)\frac{1}{\sqrt{|J|}}\mathbf{1}_J(\xi)\,d\xi \\ \approx\ &\left(\frac{1}{\operatorname{dist}(0,J)}\right)^{\frac{n-1}{2}}\frac{1}{\sqrt{|I|\,|J|}}|J|=\left(\frac{1}{\ell(I)^2\operatorname{dist}(0,J)}\right)^{\frac{n-1}{2}}\sqrt{|I|\,|J|}\lesssim 2^{-d\frac{n-1}{2}}\sqrt{|I|\,|J|}. \end{aligned}$$

The intermediate terms $\mathfrak{P}_{\varphi_I^{\eta},\phi}^{(\ell)}(y,\lambda)$ can be estimated in a similar way.

Next we estimate the inner product with the error term $\mathfrak{R}^{(M+1)}_{h^{n-1,\eta}_{I;\kappa},\phi}$ using the bound (3.13),

$$\left|\mathfrak{R}^{(M+1)}_{h^{n-1,\eta}_{I;\kappa},\phi}(y,\lambda)\right| \leq C_M \lambda^{-\frac{n-1}{2}-M-1}\left\|\left(\mathrm{Id}-\triangle_x\right)^N h^{n-1,\eta}_{I;\kappa}\right\|_{L^1\left(\mathbb{R}^{n-1}_x\right)\times L^\infty\left(\mathbb{R}^n_y-1\right)} \leq C_M \lambda^{-\frac{n-1}{2}-M-1}\frac{1}{\ell(I)^{2N}}\sqrt{|I|},$$

for $N > 1+\frac{n-1}{2}$, to obtain

$$\left|\left\langle \mathfrak{R}^{(M+1)}_{h^{n-1,\eta}_{I;\kappa},\phi}, h^{n,\eta}_{J;\kappa}\right\rangle\right| = \left|\int_{\mathbb{R}^n}\mathfrak{R}^{(M+1)}_{h^{n-1,\eta}_{I;\kappa},\phi}(\xi)\, h^{n,\eta}_{J;\kappa}(\xi)\, d\xi\right| \tag{4.18}$$

$$\lesssim \left(\frac{1}{\operatorname{dist}(0,J)}\right)^{\frac{n-1}{2}+1}\left(\frac{1}{\ell(I)^2}\right)^N\sqrt{|I|\,|J|} \approx 2^{-d\left(\frac{n-1}{2}+1\right)}\left(\frac{1}{\ell(I)^2}\right)^\tau\sqrt{|I|\,|J|},$$

where $\tau = N-\frac{n+1}{2} > 0$.

Adding these estimates gives,

$$\left|\left\langle Th^{n-1,\eta}_{I;\kappa}, h^{n,\eta}_{J;\kappa}\right\rangle\right| \lesssim \left\{\sum_{\ell=0}^M 2^{-d\left(\frac{n-1}{2}+\ell\right)} + 2^{-d\left(\frac{n-1}{2}+M+1\right)}\left(\frac{1}{\ell(I)^2}\right)^\tau\right\}\sqrt{|I|\,|J|},$$

which completes the proof of (4.15). Since $N-\frac{n+1}{2}\in\frac{1}{2}\mathbb{Z}$, we may assume $0<\tau\leq 1$.

4.2.4. *Tangential integration by parts.* Finally, we improve on the crude estimate (4.5) in the case $k=0$, $d\geq 0$ and $m\in\mathbb{N}$ using a tangential integration by parts as our last principle of decay, where the supports of I and $\Phi^{-1}(\pi_{\tan}J)$ are separated by at least $\ell(I)$. Let $(I,J)\in\mathcal{P}^{0,d}_m$ with $d\geq 0$, i.e.

$$\operatorname{dist}(\pi_{\tan}J, I)\approx 2^m\ell(I), \quad \ell(J)=1, \quad \text{and } \frac{2^{d-1}}{\ell(I)^2}\leq \operatorname{dist}(0,J)\leq\frac{2^{d+1}}{\ell(I)^2}.$$

Recall again the change of variable in (3.6) and (3.7),

$$\begin{aligned}\left\langle Th^{n-1,\eta}_{I;\kappa}, h^{n,\eta}_{J;\kappa}\right\rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^{n-1}} e^{-i\Phi(x)\cdot\xi} h^{n-1,\eta}_{I;\kappa}(x)\, h^{n,\eta}_{J;\kappa}(\xi)\, dxd\xi\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} e^{-i\lambda\phi(x,y)}\varphi^\eta_I(x)\,\widetilde{\psi}^\eta_J(y,\lambda)\, dxdyd\lambda,\end{aligned}$$

where

$$\begin{aligned}\phi(x,y) &\equiv \Phi(x)\cdot\Phi(y),\\ \varphi^\eta_I(x) &\equiv h^{n-1,\eta}_{I;\kappa}(x) \text{ and } \psi^\eta_J(\xi) = h^{n,\eta}_{J;\kappa}(\xi),\\ \widetilde{\psi}^\eta_J(y,\lambda) &\equiv \psi^\eta_J\left(\lambda y, \lambda\sqrt{1-|y|^2}\right)\frac{\lambda^{n-1}}{\sqrt{1-|y|^2}}.\end{aligned}$$

Here the supports of $\pi_{\tan}J$ and I are separated by a distance of approximately $2^m\ell(I)$, and $\ell(\pi_{\tan}J)\lesssim\ell(I)$, and this suggests we should integrate by parts in the variables x and y.

So let $y_J=\Phi^{-1}(\pi_{\tan}c_J)$ and $\mathbf{v}=\frac{y_J-c_I}{|y_J-c_I|}\in\mathbb{S}^{n-2}$ be the unit vector in the direction of y_J-c_I, which is close to the direction of $y-x$ for $x\in I$ and $y=\Phi^{-1}(\pi_{\tan}\xi)$ with $\xi\in J$. Consider the directional partial derivative $D^x_{\mathbf{v}}=\mathbf{v}\cdot\frac{\partial}{\partial x}$, and note that

$$D^x_{\mathbf{v}}\phi(x,y) = (D_{\mathbf{v}}\Phi)(x)\cdot\Phi(y).$$

Since $(D_{\mathbf{v}}\Phi)(x)$ is perpendicular to $\Phi(x)$ in $\mathbb{R}^n$, we have the estimate

$$|D^x_{\mathbf{v}}\phi(x,y)|\approx|x-y|\ , \qquad x\in I,\xi\in J.$$

Now we compute

$$D^x_{\mathbf{v}}e^{-i\lambda\phi(x,y)} = -i\lambda e^{-i\lambda\phi(x,y)}D^x_{\mathbf{v}}\phi(x,y) = -i\lambda e^{-i\lambda\phi(x,y)}(D_{\mathbf{v}}\Phi)(x)\cdot\Phi(y),$$

and so

$$\left(\frac{1}{-i\lambda(D_{\mathbf{v}}\Phi)(x)\cdot\Phi(y)}D^x_{\mathbf{v}}\right)^N e^{-i\lambda\phi(x,y)} = e^{-i\lambda\phi(x,y)},$$

which gives,

$$
\begin{aligned}
(4.19)\qquad & \left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\left\{\left(\frac{1}{-i\lambda\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)}D_{\mathbf{v}}^{x}\right)^{N}e^{i\lambda\phi(x,y)}\right\}\varphi_{I}^{\eta}(x)\,\widetilde{\psi}_{J}^{\eta}(y,\lambda)\,dxdyd\lambda \\
&= i^{N}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}e^{i\lambda\phi(x,y)}\left\{\left(D_{\mathbf{v}}^{x}\frac{1}{\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)}\right)^{N}\right\}\varphi_{I}^{\eta}(x)\,\widetilde{\psi}_{J}^{\eta}(y,\lambda)\,dxdy\frac{d\lambda}{\lambda^{N}}.
\end{aligned}
$$

This integral can be estimated by

$$
\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\left|\left(D_{\mathbf{v}}^{x}\frac{1}{\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)}\right)^{N}\varphi_{I}^{\eta}(x)\right|\frac{1}{\lambda^{N}}\left|\widetilde{\psi}_{J}^{\eta}(y,\lambda)\right|dxdyd\lambda,
$$

where we have the following pointwise estimates for $N=0$ and $N=1$,

$$
\begin{aligned}
& \left|\varphi_{I}^{\eta}(x)\right|\lesssim\frac{1}{\sqrt{|I|}}, \\
\text{and}\qquad & \left|D_{\mathbf{v}}^{x}\frac{1}{\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)}\varphi_{I}^{\eta}(x)\right|\frac{1}{\lambda}\lesssim\frac{\left|\partial_{x}\varphi_{I}^{\eta}(x)\right|}{\lambda\left|\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)\right|}+\frac{\left|\varphi_{I}^{\eta}(x)\right|\left|\left(D_{\mathbf{v}}^{2}\Phi\right)(x)\cdot\Phi(y)\right|}{\lambda\left|\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)\right|^{2}} \\
\lesssim\ & \frac{\frac{1}{\eta\ell(I)}\frac{1}{\sqrt{|I|}}}{\lambda|x-y|}+\frac{\frac{1}{\sqrt{|I|}}}{\lambda|x-y|^{2}}\lesssim\frac{\frac{1}{\eta}\frac{1}{\sqrt{|I|}}}{\lambda 2^{m}\ell(I)\,\ell(I)}+\frac{\frac{1}{\sqrt{|I|}}}{\lambda\left(2^{m}\ell(I)\right)^{2}} \\
\lesssim\ & \frac{1}{\lambda 2^{m}\ell(I)^{2}}\frac{1}{\sqrt{|I|}}=2^{-m}\frac{1}{\operatorname{dist}(0,J)\,\ell(I)^{2}}\frac{1}{\sqrt{|I|}}.
\end{aligned}
$$

We claim that by induction on N we have

$$
(4.20)\qquad \frac{1}{\lambda^{N}}\left|\left(D_{\mathbf{v}}^{x}\frac{1}{\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)}\right)^{N}\varphi_{I}^{\eta}(x)\right|\lesssim 2^{-Nm}\left(\frac{1}{\operatorname{dist}(0,J)\,\ell(I)^{2}}\right)^{N}\frac{1}{\sqrt{|I|}}.
$$

For simplicity, we illustrate the inductive step in the case $N=2$, and compute

$$
\begin{aligned}
& D_{\mathbf{v}}^{x}\frac{1}{\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)}D_{\mathbf{v}}^{x}\frac{1}{\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)}\varphi_{I}^{\eta}(x) \\
&= D_{\mathbf{v}}^{x}\left(\frac{D_{\mathbf{v}}^{x}\varphi_{I}^{\eta}(x)}{\left[\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)\right]^{2}}-\frac{\varphi_{I}^{\eta}(x)\left(D_{\mathbf{v}}^{2}\Phi\right)(x)\cdot\Phi(y)}{\left[\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)\right]^{3}}\right) \\
&= \frac{\left(D_{\mathbf{v}}^{x}\right)^{2}\varphi_{I}^{\eta}(x)}{\left[\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)\right]^{2}}-3\frac{D_{\mathbf{v}}^{x}\varphi_{I}^{\eta}(x)\left(D_{\mathbf{v}}^{2}\Phi\right)(x)\cdot\Phi(y)}{\left[\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)\right]^{3}} \\
&\quad -\frac{\varphi_{I}^{\eta}(x)\left(D_{\mathbf{v}}^{3}\Phi\right)(x)\cdot\Phi(y)}{\left[\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)\right]^{3}}+3\frac{\varphi_{I}^{\eta}(x)\left[\left(D_{\mathbf{v}}^{2}\Phi\right)(x)\cdot\Phi(y)\right]^{2}}{\left[\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)\right]^{4}},
\end{aligned}
$$

which gives,

$$
\begin{aligned}
\frac{1}{\lambda^{2}}\left|\left(D_{\mathbf{v}}^{x}\frac{1}{\left(D_{\mathbf{v}}\Phi\right)(x)\cdot\Phi(y)}\right)^{2}\varphi_{I}^{\eta}(x)\right| &\lesssim \frac{1}{\lambda^{2}}\left(\frac{\left(\frac{1}{\eta\ell(I)}\right)^{2}\frac{1}{\sqrt{|I|}}}{|x-y|^{2}}+\frac{\left(\frac{1}{\eta\ell(I)}\right)\frac{1}{\sqrt{|I|}}}{|x-y|^{3}}+\frac{\frac{1}{\sqrt{|I|}}|x-y|}{|x-y|^{3}}+\frac{\frac{1}{\sqrt{|I|}}}{|x-y|^{4}}\right) \\
&\lesssim \frac{1}{\lambda^{2}}\left(\frac{1}{2^{2m}\ell(I)^{4}}+\frac{1}{2^{3m}\ell(I)^{4}}+\frac{1}{2^{4m}\ell(I)^{4}}\right)\frac{1}{\sqrt{|I|}} \\
&\lesssim \frac{1}{\lambda^{2}}\frac{1}{2^{2m}\ell(I)^{4}}\frac{1}{\sqrt{|I|}}=2^{-2m}\left(\frac{1}{\operatorname{dist}(0,J)\,\ell(I)^{2}}\right)^{2}\frac{1}{\sqrt{|I|}},
\end{aligned}
$$

which is the case $N=2$ of (4.20). The general case is similar.

The estimate (4.20) leads to the inner product estimate,

$$\begin{aligned}
(4.21)\qquad & \left|\left\langle Th^{n-1,\eta}_{I;\kappa}, h^{n,\eta}_{J;\kappa}\right\rangle\right| \\
\leq\ & \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{1}{\lambda^N}\left|\left\{\left(D^x_{\mathbf{v}}\frac{1}{(D_{\mathbf{v}}\Phi)(x)\cdot\Phi(y)}\right)^N\right\}\varphi^\eta_I(x)\right|\left|\widehat{\psi}^\eta_J(y,\lambda)\right| dxdyd\lambda \\
\leq\ & \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} 2^{-Nm}\left(\frac{1}{\operatorname{dist}(0,J)\,\ell(I)^2}\right)^N \frac{1}{\sqrt{|I|}}\left|\widehat{\psi}^\eta_J(y,\lambda)\right| dxdyd\lambda \\
\leq\ & 2^{-Nm}\left(\frac{1}{\operatorname{dist}(0,J)\,\ell(I)^2}\right)^N \frac{1}{\sqrt{|I|}}\,|I|\left\|\widehat{\psi}^\eta_J(y,\lambda)\right\|_{L^1} \approx 2^{-N(m+d)}\sqrt{|I|\,|J|},
\end{aligned}$$

since $\operatorname{dist}(0,J)\,\ell(I)^2 \approx 2^d$ for $(I,J)\in\mathcal{P}^{0,d}_m$, $d\geq 0$.

5. Interpolation estimates

Here we describe the decay principle needed to handle sums of resonant inner products by probability. In fact the probabilistic estimates here rely only on the *transversality* induced by the curvature of the sphere, and not on stationary phase estimates. Throughout this subsection we will use the familiar notation $\widehat{\varphi}$ for the Fourier transform of φ, and we will use the parameter $s\in\mathbb{N}$ to pigeonhole the side length 2^{-s} of a cube $I\in\mathcal{G}$. Let

$$\mathsf{Q}^s_U \equiv \sum_{I\in\mathcal{G}_s[U]} \triangle^{n-1}_{I;\kappa}, \qquad \text{where } \mathcal{G}_s[U]=\left\{I\in\mathcal{G}: I\subset U \text{ and } \ell(I)=2^{-s}\right\},$$

be the Alpert projection onto $\mathcal{G}_s[U]$, i.e. $\triangle_{I;\kappa}$ and $\triangle^\eta_{I;\kappa}$ are restricted to dyadic subcubes I of U at depth s in the grid $\mathcal{G}$. Then we have

$$\begin{aligned}
\left(\mathsf{Q}^s_U\right)^{\spadesuit} f &= S_{\kappa,\eta}\mathsf{Q}^s_U\left(S_{\kappa,\eta}\right)^{-1} f = S_{\kappa,\eta}\sum_{I\in\mathcal{G}_s[U]}\left\langle \left(S_{\kappa,\eta}\right)^{-1} f, h^{n-1}_{I;\kappa}\right\rangle h^{n-1}_{I;\kappa} \\
&= \sum_{I\in\mathcal{G}_s[U]}\left\langle \left(S_{\kappa,\eta}\right)^{-1} f, h^{n-1}_{I;\kappa}\right\rangle h^{n-1,\eta}_{I;\kappa} = \sum_{I\in\mathcal{G}_s[U]}\triangle^{n-1,\eta}_{I;\kappa} f.
\end{aligned}$$

Let $\varphi\in C^\infty(\mathbb{R}^n)$ be a smooth nonnegative function satisfying

$$\varphi(\xi) = \begin{cases} 1 & \text{if} \quad \xi\in B_{\mathbb{R}^n}(0,1) \\ 0 & \text{if} \quad \xi\notin B_{\mathbb{R}^n}(0,2) \end{cases}, \tag{5.1}$$

and set

$$\varphi_t(\xi) = 2^{-tn}\varphi\left(2^{-t}\xi\right), \qquad \text{for } t\geq 0,$$

where we note that the scaling is with respect to 2^{-t} instead of the usual scaling t. Recall that $\Phi(x) = \left(x,\sqrt{|x|^2}\right)\in\mathbb{S}^{n-1}$ for $x\in S$. Define the spherical measure f^I_Φ by

$$\begin{aligned}
(5.2)\qquad f^I_\Phi(z) &\equiv \Phi_*\,\triangle^{n-1,\eta}_{I;\kappa} f = \triangle^{n-1,\eta}_{I;\kappa} f\left(\Phi^{-1}(z)\right)\det\partial\Phi^{-1}(z)\,d\sigma_{n-1}(z) \\
&= \left\langle \left(S_{\kappa,\eta}\right)^{-1} f, h^{n-1}_{I;\kappa}\right\rangle h^{n-1,\eta}_{I;\kappa}\left(\Phi^{-1}(z)\right)\det\partial\Phi^{-1}(z)\,d\sigma_{n-1}(z),
\end{aligned}$$

and set

$$f^s_\Phi(z) \equiv \sum_{I\in\mathcal{G}_s[U]} f^I_\Phi(z) = \Phi_*\sum_{I\in\mathcal{G}_s[U]} f^I(z) = \Phi_*\left(\mathsf{Q}^s_U\right)^{\spadesuit} f\ .$$

Note that the spherical measure f^I_Φ has mass roughly $\left|\left\langle \left(S_{\kappa,\eta}\right)^{-1} f, h^{n-1}_{I;\kappa}\right\rangle\right| 2^{-s(n-1)}$ for $I\in\mathcal{G}_s[U]$ and is supported in $\mathbb{S}^{n-1}$.

Here is the model result of this subsection, where we recall that

$$\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}^s_U\right)^{\spadesuit} f = S_{\kappa,\eta}\mathcal{A}_{\mathbf{a}}\mathsf{Q}^s_U\left(S_{\kappa,\eta}\right)^{-1} f = \sum_{I\in\mathcal{G}_s[U]} a_I\,\triangle^{n-1,\eta}_{I;\kappa} f.$$

Proposition 32. *Let $n \geq 2$. Then for $p > \frac{2n}{n-1}$, there is $\varepsilon_{p,n} > 0$ such that for every $s \in \mathbb{N}$, and every $f \in L^p\left(\mathbb{R}^{n-1}\right)$, we have,*

$$\left(\mathbb{E}^{\mu}_{2^{\mathcal{G}}} \left\| T\left[\left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}^s_U\right)^{\spadesuit} f\right]\right\|^p_{L^p(B_n(0,2^s))}\right)^{\frac{1}{p}} \lesssim 2^{-s\varepsilon_{p,n}} \|f\|_{L^p(\mathbb{R}^{n-1})}, \tag{5.3}$$

where the implied constant depends on n, p and U, but is independent of $s \in \mathbb{N}$.

This estimate is a building block toward controlling the resonant portion of the disjoint form, which however requires a much larger localization to a ball of radius 2^{2s}.

We prove Proposition 32 in three steps, beginning with Plancherel's theorem in the form of a lemma that allows improvement of the traditional L^2 and L^4 curvature estimates in the presence of probability and Alpert wavelets. Then we use the scaled Marcinkiewicz interpolation theorem to obtain the desired conclusion if certain L^2 and L^4 estimates hold. Finally we establish these L^2 and L^4 estimates to complete the proof of Proposition 32.

Recall that

$$f^s_\Phi \equiv \Phi_* \left(\mathsf{Q}^s_U\right)^{\spadesuit} f \text{ and } f^I_\Phi \equiv \left(\triangle^{n-1,\eta}_{I;\kappa} f\right)_\Phi . \tag{5.4}$$

For $s \leq r \leq 2s$, define a fattened n-dimensional measure $f^s_{\Phi,r}$ by

$$f^s_{\Phi,r} \equiv f^s_\Phi * \varphi_r = \sum_{I \in \mathcal{G}_s[U]} f^I_\Phi * \varphi_r = \sum_{I \in \mathcal{G}_s[U]} f^I_{\Phi,r}, \qquad \text{where } f^I_{\Phi,r} \equiv f^I_\Phi * \varphi_r . \tag{5.5}$$

We will use the upper majorant properties of L^2 and L^4 (we use this latter phrase loosely to denote that convolution is a positive operation) to obtain Lemma 33 below in order to significantly reduce the norm $\left\|T\left(\mathsf{Q}^s_U\right)^{\spadesuit} f\right\|^p_{L^p\left(|\widehat{\varphi_s}|^4 \lambda_n\right)}$ when averaged over involutive Alpert multipliers of f.

Note: The n-dimensional measure $f^I_{\Phi,r} = f^I_\Phi * \varphi_r$ is supported in the fattened spherical cap

$$\mathcal{I}_{2^{-r}} \equiv \left\{z \in \mathbb{R}^n : \text{dist}\left(z, \text{Supp}\, f^I_\Phi\right) \lesssim 2^{-r}\right\},$$

which for $r = 2s$ is roughly a rectangular block of side lengths $2^{-2s} \times 2^{-s}$ oriented perpendicular to a normal of the spherical cap $\text{Supp}\, f^I_\Phi$. We have the estimate,

$$\left|f^I_{\Phi,r}(z)\right| \lesssim \left|\left\langle S^{-1}_{\kappa,\eta} f, h^{n-1}_{I;\kappa}\right\rangle\right| 2^r 2^{s\frac{n-1}{2}} \mathbf{1}_{\mathcal{I}_{2^{-r}}}(z) . \tag{5.6}$$

Lemma 33. *Suppose $s \in \mathbb{N}$, and φ is as in (5.1) above, so that $|\widehat{\varphi_s}| \approx 1$ on $B(0, C2^s)$. Then for $s \leq r \leq 2s$, we have*

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|\widehat{f^s_\Phi}(\xi)\right|^2 |\widehat{\varphi_{2s}}(\xi)|^2 |\widehat{\varphi_r}(\xi)|^2 d\xi &= \int_{\mathbb{R}^n} \left|\widehat{f^s_{\Phi,2s}}(\xi)\right|^2 |\widehat{\varphi_r}(\xi)|^2 d\xi, \\
\int_{\mathbb{R}^n} \left|\widehat{f^s_\Phi}(\xi)\right|^4 |\widehat{\varphi_r}(\xi)|^4 d\xi &= \int_{\mathbb{R}^n} \left|\widehat{f^s_{\Phi,r}}(\xi)\right|^4 d\xi,
\end{aligned}$$

Proof. From Plancherel's formula, we have

$$\int_{\mathbb{R}^n} \left|\widehat{f^s_\Phi}(\xi)\right|^2 |\widehat{\varphi_{2s}}(\xi)|^2 |\widehat{\varphi_r}(\xi)|^4 d\xi = \int_{\mathbb{R}^n} \left|\widehat{f^s_\Phi * \varphi_{2s}}(\xi)\right|^2 |\widehat{\varphi_r}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} \left|\widehat{f^s_{\Phi,2s}}(\xi)\right|^2 |\widehat{\varphi_r}(\xi)|^2 d\xi,$$

and using Plancherel's formula again with the convolution identity $\widehat{F * G} = \widehat{F}\widehat{G}$, gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|\widehat{f^s_\Phi}(\xi)\right|^4 |\widehat{\varphi_r}(\xi)|^4 d\xi = \int_{\mathbb{R}^n} \left|f^s_\Phi * \widehat{f^s_\Phi * \varphi_r} * \varphi_r(\xi)\right|^2 d\xi \\
= &\int_{\mathbb{R}^n} \overline{f^s_\Phi * \widehat{f^s_\Phi * \varphi_r} * \varphi_r(\xi)}(\xi) \;\; f^s_\Phi * \widehat{f^s_\Phi * \varphi_r} * \varphi_r(\xi)\, d\xi \\
= &\int_S \overline{f^s_{\Phi,r} * f^s_{\Phi,r}(x)}\; f^s_{\Phi,r} * f^s_{\Phi,r}(x)\;\, dx = \int_{\mathbb{R}^n} \left|\widehat{f^s_{\Phi,r}}(\xi)\right|^4 d\xi.
\end{aligned}$$

□

Here is the lemma that obtains the required L^p bounds from improved L^2 and L^4 bounds.

Lemma 34. *Let $n \geq 2$ and $s \in \mathbb{N}$. Assume that*

$$\left\| \widehat{f^s_{\Phi,2s}} \right\|_{L^2\left(|\widehat{\varphi_s}|^2 \lambda_n\right)} \lesssim 2^{\frac{s}{2}} \|f\|_{L^2(S)}, \tag{5.7}$$

$$\left(\mathbb{E}^{\mu}_{2^{\mathcal{G}}} \left\| \left[\widehat{(\mathcal{A}_{\mathbf{a}} \mathsf{Q}^s_U)^{\spadesuit}} \right]_{\Phi,2s} \right\|^4_{L^4(\lambda_n)} \right)^{\frac{1}{4}} \lesssim 2^{-s\frac{n-2}{4}} \|f\|_{L^4(S)}.$$

Then for $p > \frac{2n}{n-1}$, there is $\varepsilon_{p,n} > 0$ such that

$$\left(\mathbb{E}^{\mu}_{2^{\mathcal{G}}} \left\| \left[\widehat{(\mathcal{A}_{\mathbf{a}} \mathsf{Q}^s_U)^{\spadesuit}} \right]_{\Phi,2s} \right\|^p_{L^p\left(|\widehat{\varphi_s}|^2 |\widehat{\varphi_{2s}}|^4 \lambda_n\right)} \right)^{\frac{1}{p}} \lesssim 2^{-s\varepsilon_{p,n}} \|f\|_{L^p(\mathbb{R}^{n-1})},$$

holds for every $s \in \mathbb{N}$ with implied constant independent of Ψ and s.

Note in particular that Lemma 34 implies (5.3) in Proposition 32 .

Proof. Combining Lemma 33 with the assumptions (5.7) gives the pair of inequalities,

$$\left\| T \left(\mathsf{Q}^s_U\right)^{\spadesuit} f \right\|_{L^2\left(|\widehat{\varphi_s}|^2 |\widehat{\varphi_{2s}}|^4 \lambda_n\right)} \lesssim 2^{\frac{s}{2}} \|f\|_{L^2(S)},$$

$$\left(\mathbb{E}^{\mu}_{2^{\mathcal{G}}} \left\| T \left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}^s_U\right)^{\spadesuit} f \right\|^4_{L^4\left(|\widehat{\varphi_s}|^2 |\widehat{\varphi_{2s}}|^4 \lambda_n\right)} \right)^{\frac{1}{4}} \lesssim 2^{-s\frac{n-2}{4}} \|f\|_{L^4(S)}.$$

Indeed,

$$\begin{aligned}
&\left\| T \left(\mathsf{Q}^s_U\right)^{\spadesuit} f \right\|^2_{L^2\left(|\widehat{\varphi_s}|^2 |\widehat{\varphi_{2s}}|^4 \lambda_n\right)} \leq \left\| T \left(\mathsf{Q}^s_U\right)^{\spadesuit} f \right\|^2_{L^2\left(|\widehat{\varphi_s}|^2 |\widehat{\varphi_{2s}}|^2 \lambda_n\right)} \\
=& \int_{\mathbb{R}^n} \left| T \left(\mathsf{Q}^s_U\right)^{\spadesuit} f(\xi) \right|^2 |\widehat{\varphi_s}(\xi)|^2 |\widehat{\varphi_{2s}}(\xi)|^2 d\xi \\
=& \int_{\mathbb{R}^n} \left| \left[\widehat{(\mathsf{Q}^s_U)^{\spadesuit} f} \right]_{\Phi} (\xi) \right|^2 |\widehat{\varphi_{2s}}(\xi)|^2 |\widehat{\varphi_s}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} \left| \left[\widehat{(\mathsf{Q}^s_U)^{\spadesuit} f} \right]_{\Phi,2s} (\xi) \right|^2 |\widehat{\varphi_s}(\xi)|^2 d\xi \\
=& \left\| \left[\widehat{(\mathsf{Q}^s_U)^{\spadesuit} f} \right]_{\Phi,2s} \right\|^2_{L^2\left(|\widehat{\varphi_s}|^2 \lambda_n\right)} \lesssim 2^s \left\| (\mathsf{Q}^s_U)^{\spadesuit} f \right\|^2_{L^2(S)} \lesssim 2^s \|f\|^2_{L^2(S)},
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}^{\mu}_{2^{\mathcal{G}}} \left\| T \left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}^s_U\right)^{\spadesuit} f \right\|^4_{L^4\left(|\widehat{\varphi_s}|^2 |\widehat{\varphi_{2s}}|^4 \lambda_n\right)} \leq \mathbb{E}^{\mu}_{2^{\mathcal{G}}} \left\| T \left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}^s_U\right)^{\spadesuit} f \right\|^4_{L^4\left(|\widehat{\varphi_s}|^2 |\widehat{\varphi_{2s}}|^4 \lambda_n\right)} \\
\leq& \ \mathbb{E}^{\mu}_{2^{\mathcal{G}}} \int_{\mathbb{R}^n} \left| \left(\widehat{(\mathcal{A}_{\mathbf{a}} \mathsf{Q}^s_U)^{\spadesuit} f} \right)_{\Phi} (\xi) \right|^4 |\widehat{\varphi_{2s}}(\xi)|^4 d\xi = \mathbb{E}^{\mu}_{2^{\mathcal{G}}} \int_{\mathbb{R}^n} \left| \sum_{I \in \mathcal{G}_s[U]} a_I \left(\widehat{\triangle^{n-1,\eta}_{I;\kappa} f} \right)_{\Phi} (\xi) \right|^4 |\widehat{\varphi_{2s}}(\xi)|^4 d\xi \\
=& \ \mathbb{E}^{\mu}_{2^{\mathcal{G}}} \int_{\mathbb{R}^n} \left| \left(\widehat{\sum_{I \in \mathcal{G}_s[U]} a_I \triangle^{n-1,\eta}_{I;\kappa} f} \right)_{\Phi,2s} (\xi) \right|^4 d\xi = \mathbb{E}^{\mu}_{2^{\mathcal{G}_s[U]}} \int_{\mathbb{R}^n} \left| \left(\widehat{\sum_{I \in \mathcal{G}_s[U]} a_I \left(\triangle^{n-1,\eta}_{I;\kappa} f \right)} \right)_{\Phi,2s} (\xi) \right|^4 d\xi \\
=& \ \mathbb{E}^{\mu_m}_{2^{\mathcal{G}_s[U]}} \int_{\mathbb{R}^n} \left| \sum_{I \in \mathcal{G}_s[U]} a_I \left(\widehat{\triangle^{n-1,\eta}_{I;\kappa} f} \right)_{\Phi,2s} (\xi) \right|^4 d\xi = \mathbb{E}^{\mu_m}_{2^{\mathcal{G}_s[U]}} \int_{\mathbb{R}^n} \left| \left[(\mathcal{A}_{\mathbf{a}} \mathsf{Q}^s_U)^{\spadesuit} f \right]_{\Phi,2s} (\xi) \right|^{44} d\xi \\
=& \ \mathbb{E}^{\mu_m}_{2^{\mathcal{G}_s[U]}} \left\| \left[(\mathcal{A}_{\mathbf{a}} \mathsf{Q}^s_U)^{\spadesuit} f \right]_{\Phi,2s} \right\|^4_{L^4(\lambda_n)} \lesssim 2^{-s(n-2)} \left\| (\mathsf{Q}^s_U)^{\spadesuit} f \right\|^4_{L^4(\lambda_{n-1})} \lesssim 2^{-s(n-2)} \|f\|^4_{L^4(\lambda_{n-1})},
\end{aligned}$$

since all three operators in the factorization $(\mathsf{Q}^s_U)^{\spadesuit} = S_{\kappa,\eta} \mathsf{Q}^s_U (S_{\kappa,\eta})^{-1}$ are bounded on $L^4(\lambda_{n-1})$.

These L^2 and L^4 estimates can be recast in terms of square functions by Khintchine's inequalities, and we will now show that the scaled Marcinkiewicz interpolation theorem applies to obtain (5.3).

Indeed, by Khinchine's inequalities, the above bounds are equivalent to

$$\begin{aligned} \left\| \mathcal{S}_{T,s} f \right\|_{L^2(\lambda_n)} &\lesssim 2^{\frac{s}{2}} \left\| f \right\|_{L^2(\sigma_{n-1})} , \\ \left\| \mathcal{S}_{T,s} f \right\|_{L^4\left(\mathbf{1}_{B(0,2^s)}\lambda_n\right)} &\lesssim 2^{-s\frac{n-2}{4}} \left\| f \right\|_{L^4(\sigma_{n-1})} , \end{aligned}$$

where $\mathcal{S}_{T,s}$ is the square function defined by

$$\mathcal{S}_{T,s} f \equiv \left(\sum_{I \in \mathcal{G}_s[U]} \left| T_S \bigtriangleup_{I;\kappa}^{n-1,\eta} f \right|^2 \right)^{\frac{1}{2}} .$$

The sublinear operator $\mathcal{S}_{T,s}$ is actually *linearizable* since it is the supremum of the linear operators $L_{\mathbf{u}} f \equiv T_S \sum_{I \in \mathcal{G}_s[U]} u_I \bigtriangleup_{I;\kappa}^{n-1,\eta} f$ taken over all vectors $\mathbf{u} = (u_I)_{I \in \mathcal{G}_s[S]}$ with $|\mathbf{u}|_{\ell^2} = 1$. Then by the scaled Marcinkiewicz theorem applied to $\mathcal{S}_{T,s}$, see e.g. [Tao2, Remark 29], we have

$$\left\| \mathcal{S}_{T,s} f \right\|_{L^p} \leq C_{n,p} 2^{\frac{s}{2}(1-\theta)} 2^{-s\frac{n-2}{4}\theta} = C_{n,p} 2^{\frac{s}{2}\left(1-\left(2-\frac{4}{p}\right)\right)} 2^{-s\frac{n-2}{4}\left(2-\frac{4}{p}\right)} = C_{n,p} 2^{-s\varepsilon_{n,p}},$$

where

$$\varepsilon_{n,p} = \frac{n-2}{4}\left(2-\frac{4}{p}\right) - \frac{1}{2}\left(1-\left(2-\frac{4}{p}\right)\right) = \frac{n-1}{2p}\left(p-\frac{2n}{n-1}\right) > 0,$$

for $p > \frac{2n}{n-1}$. Another application of Khintchine's inquality converts this bound back to the expectation bound,

$$\left(\mathbb{E}_{2^{\mathcal{G}_s[U]}}^{\mu_s} \left\| T \left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s \right)^{\spadesuit} f \right\|_{L^p(B_n(0,2^s))}^p \right)^{\frac{1}{p}} \lesssim C_{n,p} 2^{-s\varepsilon_{n,p}} \left\| f \right\|_{L^p(\mathbb{R}^{n-1})} .$$

Thus we have

$$\mathbb{E}_{2^{\mathcal{G}_s[U]}}^{\mu_s} \left\| T \left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s \right)^{\spadesuit} f(\xi) \right\|_{L^p(B_n(0,2^s))}^p \lesssim 2^{-sp\varepsilon_{n,p}} \left\| f \right\|_{L^p(\mathbb{R}^{n-1})}^p ,$$

which completes the proof of Lemma 34. □

It remains to establish the improved bounds in (5.7), which we accomplish in the next two subsections. Once this is done, the proof of Proposition 32 is complete.

5.1. **The L^2 estimate.** We first compute the norm of $\Lambda_{\mathsf{Q}_U^s}^{2s}$ from $L^2(\lambda_{n-1})$ to $L^2\left(\left|\widehat{\varphi_s}\right|^2 \lambda_n\right)$, where

$$\Lambda_{\mathsf{Q}_U^s}^{2s} f \equiv \left(\widehat{(\mathsf{Q}_U^s)^{\spadesuit} f} \right)_{\Phi,2s} .$$

We write $f_U^s \equiv (\mathsf{Q}_U^s)^{\spadesuit} f$ for convenience in notation so that we have,

$$\begin{aligned} &\left\| \Lambda_{\mathsf{Q}_U^s}^{2s} f \right\|_{L^2\left(\left|\widehat{\varphi_s}\right|^2\lambda_n\right)}^2 = \int_{\mathbb{R}^n} \left| \widehat{(f_U^s)_{\Phi,2s}}(\xi) \right|^2 \left| \widehat{\varphi_s}(\xi) \right|^2 d\xi \\ = &\int_{\mathbb{R}^n} \overline{\widehat{(f_U^s)_{\Phi,2s}} * \varphi_s(\xi)} \; \widehat{(f_U^s)_{\Phi,2s}} * \varphi_s(\xi) \, d\xi \\ = &\sum_{I,K \in \mathcal{G}_s[U]} \int_{\mathbb{R}^n} \overline{\widehat{f_{\Phi,2s}^I * \varphi_s}(\xi)} \; \widehat{f_{\Phi,2s}^K * \varphi_s}(\xi) \, d\xi = \sum_{I,K \in \mathcal{G}_s[U]} \int_S \overline{f_{\Phi,2s}^I * \varphi_s(x)} \; \left(f_{\Phi,2s}^K * \varphi_s \right)(x) \, dx. \end{aligned}$$

Noting that the supports of $f_{\Phi,2s}^I * \varphi_s$ and $f_{\Phi,2s}^K * \varphi_s$ are essentially disjoint unless $I \sim K$, and recalling the definition of $\mathcal{I}_{2^{-s}}$ in Note 5, we can use (5.6),

$$\left| f_{\Phi,r}^I(z) \right| \lesssim \left| \left\langle S_{\kappa,\eta}^{-1} f, h_{I;\kappa}^{n-1} \right\rangle \right| 2^r 2^{s\frac{n-1}{2}} \mathbf{1}_{\mathcal{I}_{2^{-r}}}(z) ,$$

with $r=s$ to estimate the above expression by

$$\left\|\Lambda^{2s}_{\mathsf{Q}^s_U}f\right\|^2_{L^2\left(|\widehat{\varphi_s}|^2\lambda_n\right)} \lesssim \sum_{I\in\mathcal{G}_s[U]}\int_{\mathbb{R}^n}\left|f^I_{\Phi,2s}*\varphi_s(\xi)\right|^2 d\xi \tag{5.8}$$
$$\lesssim \sum_{I\in\mathcal{G}_s[U]}\int_{\mathbb{R}^n}\left|\left|\left\langle S^{-1}_{\kappa,\eta}f,h^{n-1}_{I;\kappa}\right\rangle\right|2^{2s}2^{s\frac{n-1}{2}}\mathbf{1}_{\mathcal{I}_{2^{-2s}}}*\varphi_s(\xi)\right|^2 d\xi$$
$$\lesssim \sum_{I\in\mathcal{G}_s[U]}\left|\left\langle S^{-1}_{\kappa,\eta}f,h^{n-1}_{I;\kappa}\right\rangle\right|^2\int_{\mathbb{R}^n}\left|2^s2^{s\frac{n-1}{2}}\mathbf{1}_{\mathcal{I}_{2^{-s}}}(\xi)\right|^2 d\xi,$$

where we have used the fact that the positive measures $\left|\mathbf{1}_{\mathcal{I}_{2^{-2s}}}*\varphi_s\right|$ and $2^{-s}\mathbf{1}_{\mathcal{I}_{2^{-s}}}$, are supported in roughly a common cube of side length 2^{-s}, and have roughly the same mass, i.e.

$$\int_{\mathbb{R}^n}\mathbf{1}_{\mathcal{I}_{2^{-2s}}}*\varphi_s(\xi)\,d\xi=\left(\int_{\mathbb{R}^n}\mathbf{1}_{\mathcal{I}_{2^{-2s}}}(\xi)\,d\xi\right)\left(\int_{\mathbb{R}^n}\varphi_s(\xi)\,d\xi\right)=\int_{\mathbb{R}^n}\mathbf{1}_{\mathcal{I}_{2^{-2s}}}(\xi)\,d\xi\approx 2^{-s}\int_{\mathbb{R}^n}\mathbf{1}_{\mathcal{I}_{2^{-s}}}(\xi)\,d\xi. \tag{5.9}$$

Then we continue with

$$\left\|\Lambda^{2s}_{\mathsf{Q}^s_U}f\right\|^2_{L^2\left(|\widehat{\varphi_s}|^4\lambda_n\right)} \lesssim \sum_{I\in\mathcal{G}_s[U]}\left|\left\langle S^{-1}_{\kappa,\eta}f,h^{n-1}_{I;\kappa}\right\rangle\right|^2\left(2^s2^{s\frac{n-1}{2}}\right)^2|\mathcal{I}_{2^{-s}}|$$
$$= 2^s\sum_{I\in\mathcal{G}_s[U]}\left|\left\langle S^{-1}_{\kappa,\eta}f,h^{n-1}_{I;\kappa}\right\rangle\right|^2\lesssim 2^s\left\|S^{-1}_{\kappa,\eta}f\right\|^2_{L^2(\mathbb{R}^{n-1})}\lesssim 2^s\|f\|^2_{L^2(S)}.$$

This proves the first line in (5.7).

5.2. **The probabilistic L^4 estimate.** Now we turn to computing the norm of Λ_{2s} from $L^4(\lambda_{n-1})$ to $L^4(\mathbb{R}^n)$. We have using $f^s_U\equiv(\mathsf{Q}^s_U)^{\spadesuit}f$ that

$$\|f^s_U\|^4_{L^4(\lambda_{n-1})}=\int_{\mathbb{R}^{n-1}}\left(\sum_{I\in\mathcal{G}_s[U]}\left\langle(S_{\kappa,\eta})^{-1}f,h^{n-1}_{I;\kappa}\right\rangle h^{n-1,\eta}_{I;\kappa}(x)\right)^4 dx$$
$$\approx \int_{\mathbb{R}^{n-1}}\sum_{I\in\mathcal{G}_s[U]}\left(\left\langle(S_{\kappa,\eta})^{-1}f,h^{n-1}_{I;\kappa}\right\rangle h^{n-1,\eta}_{I;\kappa}(x)\right)^4 dx$$
$$= \sum_{I\in\mathcal{G}_s[U]}\left|\left\langle(S_{\kappa,\eta})^{-1}f,h^{n-1}_{I;\kappa}\right\rangle\right|^4\int_{\mathbb{R}^{n-1}}\left|h^{n-1,\eta}_{I;\kappa}(x)\right|^4 dx$$
$$\approx \sum_{I\in\mathcal{G}_s[U]}\left|\left\langle(S_{\kappa,\eta})^{-1}f,h^{n-1}_{I;\kappa}\right\rangle\right|^4\left(\frac{1}{\sqrt{|I|}}\right)^4|I|=\sum_{I\in\mathcal{G}_s[U]}\left|\left\langle(S_{\kappa,\eta})^{-1}f,h^{n-1}_{I;\kappa}\right\rangle\right|^4\frac{1}{|I|}$$
$$= 2^{s(n-1)}\sum_{I\in\mathcal{G}_s[U]}\left|\left\langle(S_{\kappa,\eta})^{-1}f,h^{n-1}_{I;\kappa}\right\rangle\right|^4=2^{s(n-1)}\left|\breve{f}\right|^4_{\ell^4(\mathcal{G}_s[U])},$$

where $\breve{f}\equiv\left\{\left\langle(S_{\kappa,\eta})^{-1}f,h^{n-1}_{I;\kappa}\right\rangle\right\}_{I\in\mathcal{G}_s[S]}$ is the sequence of Alpert coefficients of $(S_{\kappa,\eta})^{-1}f$ restricted to $\mathcal{G}_s[S]$. Recall that $\left\|(S_{\kappa,\eta})^{-1}f\right\|_{L^p(\mathbb{R}^{n-1})}\approx\|f\|_{L^p(\mathbb{R}^{n-1})}$ by Theorem 14.

Next we calculate the $L^4(\lambda_n)$ norm of $\Lambda^{2s}_{\mathsf{Q}^s_U}f\equiv\widehat{\left((\mathsf{Q}^s_U)^{\spadesuit}f\right)}_{\Phi,2s}=\widehat{(f^s_U)}_{\Phi,2s}$:

$$\left\|\Lambda^{2s}_{\mathsf{Q}^s_U}f\right\|^4_{L^4(\lambda_n)}=\int_{\mathbb{R}^n}\left|\widehat{(f^s_U)}_{\Phi,2s}(\xi)\right|^4 d\xi=\int_{\mathbb{R}^n}\left|\sum_{I\in\mathcal{G}_s[U]}\widehat{f^I_{\Phi,2s}}(\xi)\right|^4 d\xi$$
$$= \int_{\mathbb{R}^n}\left|\sum_{I,J\in\mathcal{G}_s[U]}\widehat{f^I_{\Phi,2s}}(\xi)\,\widehat{f^J_{\Phi,2s}}(\xi)\right|^2 d\xi=\int_{\mathbb{R}^n}\left|\sum_{I,J\in\mathcal{G}_s[U]}\widehat{f^I_{\Phi,2s}*f^J_{\Phi,2s}}(\xi)\right|^2 d\xi,$$

by the Fourier convolution formula, and then by Plancherel's theorem,

$$\left\|\Lambda_{\mathsf{Q}_U^s}^{2s} f\right\|_{L^4(\lambda_n)}^4 = \int_{\mathbb{R}^n} \left| \sum_{I,J\in\mathcal{G}_s[U]} f_{\Phi,2s}^I * f_{\Phi,2s}^J(z) \right|^2 dz = \sum_{I,J,I',J'\in\mathcal{G}_s[U]} \int f_{\Phi,2s}^I * f_{\Phi,2s}^J(z)\ \ f_{\Phi,2s}^{I'} * f_{\Phi,2s}^{J'}(z)\ \ dz.$$

Now we compute the average $\mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\|\Lambda_{\mathcal{A}_{\mathbf{a}}\mathsf{Q}_U^s}^{2s} f\right\|_{L^4(\lambda_n)}^4$ over all involutive smooth Alpert multipliers $(\mathcal{A}_{\mathbf{a}}\mathsf{Q}_U^s)^{\spadesuit}$, where remembering that the functions $f_{\Phi,2s}^I$ have the η-smoothness built into their definition,

$$\begin{aligned} &\mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\|\Lambda_{\mathcal{A}_{\mathbf{a}}\mathsf{Q}_U^s}^{2s} f\right\|_{L^4(\lambda_n)}^4 \\ =\ & \mathbb{E}_{2^{\mathcal{G}}}^{\mu} \sum_{I,J,I',J'\in\mathcal{G}_s[S]} \ \sum_{(a_I,a_J,a_{I'},a_{J'})\in\{-1,1\}^{\mathcal{G}_s[U]}} \mathbb{E}_{2^{\mathcal{G}}}^{\mu} \int \left(a_I f_{\Phi,2s}^I\right) * \left(a_J f_{\Phi,2s}^J\right)(z)\ \ \left(a_{I'} f_{\Phi,2s}^{I'}\right) * \left(a_{J'} f_{\Phi,2s}^{J'}\right)(z)\ \ dz \\ =\ & 2\left\{ \sum_{\substack{I,J,I',J'\in\mathcal{G}_s[U] \\ I=J \text{ and } I'=J'}} + \sum_{\substack{I,J,I',J'\in\mathcal{G}_s[U] \\ I=I' \text{ and } J=J'}} \right\} \int f_{\Phi,2s}^I * f_{\Phi,2s}^J(z)\ \ f_{\Phi,2s}^{I'} * f_{\Phi,2s}^{J'}(z)\ \ dz \equiv \mathcal{E}_1 + \mathcal{E}_2, \end{aligned}$$

since the only summands that survive expectation are those for which $a_I a_J a_{I'} a_{J'}$ is a product of squares, i.e. the factors occur in pairs of equal sign ± 1.

Remark 35. *This is the key consequence of taking expectation, and is the only place in the paper where it arises. Note also that in $n = 2$ dimensions, Fefferman made the critical observation that the supports of the convolutions $f_{\Phi,2s}^I * f_{\Phi,2s}^J$ are essentially pairwise disjoint, so that the L^2 norm squared of the sum is the sum of the L^2 norms squared. This then led to the resolution of the extension problem in dimension $n = 2$. However, in higher dimensions this observation doesn't generalize in a simple way, since there is an $(n-2)$-dimension sphere contained inside $\mathbb{S}^{n-1}$ whose pairs of 'antipodal cubes' support functions whose convolutions all occupy the same space. The products of distinct pairs of antipodal cubes vanish under expectation, which leads to a favourable L^4 estimate.*

We have

$$\mathcal{E}_2 = 2 \sum_{I,J\in\mathcal{G}_s[U]} \int f_{\Phi,2s}^I * f_{\Phi,2s}^J(z)\ \ f_{\Phi,2s}^I * f_{\Phi,2s}^J(z)\ \ dz = 2 \sum_{I,J\in\mathcal{G}_s[U]} \int \left| f_{\Phi,2s}^I * f_{\Phi,2s}^J(z) \right|^2 dz.$$

Since the supports of $f_{\Phi,2s}^I * f_{\Phi,2s}^I$ and $f_{\Phi,2s}^{I'} * f_{\Phi,2s}^{I'}$ are disjoint unless $\operatorname{dist}(I, I') \lesssim 1$, we also have

$$\mathcal{E}_1 = 2 \sum_{I,I'\in\mathcal{G}_s[U]} \int f_{\Phi,2s}^I * f_{\Phi,2s}^I(z)\ \ f_{\Phi,2s}^{I'} * f_{\Phi,2s}^{I'}(z)\ \ dz \lesssim \sum_{I\in\mathcal{G}_s[U]} \int \left| f_{\Phi,2s}^I * f_{\Phi,2s}^I(z) \right|^2 dz.$$

Altogether we obtain

$$\begin{aligned} &\mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\|\Lambda_{\mathcal{A}_{\mathbf{a}}\mathsf{Q}_U^s}^{2s} f\right\|_{L^4(\lambda_n)}^4 \lesssim \sum_{I,J\in\mathcal{G}_s[U]} \int \left| f_{\Phi,2s}^I * f_{\Phi,2s}^J(z) \right|^2 dz \\ =\ & \sum_{I,J\in\mathcal{G}_s[U]:\ \operatorname{dist}(I,J)\lesssim 2^{-s}} \int \left| f_{\Phi,2s}^I * f_{\Phi,2s}^J(z) \right|^2 dz + \sum_{t=0}^{s} \sum_{I,J\in\mathcal{G}_s[U]:\ \operatorname{dist}(I,J)\approx 2^{-t}} \int \left| f_{\Phi,2s}^I * f_{\Phi,2s}^J(z) \right|^2 dz \\ \equiv\ & \Psi + \sum_{t=0}^{s} \Psi_t. \end{aligned}$$

Now note that the L^1 norm of $f_{\Phi,2s}^I * f_{\Phi,2s}^J$ is essentially

$$\begin{aligned} \left\| f_{\Phi,2s}^I \right\|_{L^1} \left\| f_{\Phi,2s}^J \right\|_{L^1} &\approx \left| \left\langle \left(S_{\kappa,\eta}\right)^{-1} f, h_{I;\kappa} \right\rangle \left\langle \left(S_{\kappa,\eta}\right)^{-1} f, h_{J;\kappa} \right\rangle \right| \left\|h_I\right\|_{L^1} \left\|h_J\right\|_{L^1} \\ &= \left| \left\langle \left(S_{\kappa,\eta}\right)^{-1} f, h_{I;\kappa} \right\rangle \left\langle \left(S_{\kappa,\eta}\right)^{-1} f, h_{J;\kappa} \right\rangle \right| 2^{-s(n-1)}, \end{aligned}$$

and since the volume of $R_{2s}(I,J) = \mathcal{I}_{2^{-2s}} + \mathcal{J}_{2^{-2s}}$ is essentially $2^{-sn} \operatorname{dist}(I,J)$, we have

$$\left|R_{s+t}(I,J)\right| \approx \left|R_{2s}(I,J)\right| \approx 2^{-sn} \operatorname{dist}(I,J) = 2^{-sn-t}, \qquad \text{for } \operatorname{dist}(I,J) \approx 2^{-t},$$

where the first equivalence is a simple consequence of the geometry of the situation. Thus we conclude that for $\operatorname{dist}(I,J)\approx 2^{-t}$,

$$\begin{aligned}\left\|f_{\Phi,2s}^{I}*f_{\Phi,2s}^{J}\right\|_{L^{1}} &\lesssim \left|\left\langle (S_{\kappa,\eta})^{-1}f,h_{I;\kappa}\right\rangle \left\langle (S_{\kappa,\eta})^{-1}f,h_{J;\kappa}\right\rangle\right| 2^{-s(n-1)}\\ &\approx \left\|\left|\left\langle (S_{\kappa,\eta})^{-1}f,h_{I;\kappa}\right\rangle \left\langle (S_{\kappa,\eta})^{-1}f,h_{J;\kappa}\right\rangle\right| 2^{-s(n-1)}\frac{1}{2^{-sn}\operatorname{dist}(I,J)}\mathbf{1}_{R_{2s}(I,J)}\right\|_{L^{1}}.\end{aligned}$$

Since there is $\lambda>0$ and a rectangle R_I such that $\left|f_{\Phi,2s}^{I}\right|\leq\lambda\mathbf{1}_{R_I}$ and $\left\|f_{\Phi,2s}^{I}\right\|_{L^1}\approx\left\|\lambda\mathbf{1}_{R_I}\right\|_{L^1}$, which again is a simple consequence of geometry, we then deduce the comparability of the integrands for $\operatorname{dist}(I,J)\approx 2^{-t}$,

$$\begin{aligned} f_{\Phi,2s}^{I}*f_{\Phi,2s}^{J}(z) &\approx \left|\left\langle (S_{\kappa,\eta})^{-1}f,h_{I;\kappa}\right\rangle \left\langle (S_{\kappa,\eta})^{-1}f,h_{J;\kappa}\right\rangle\right| 2^{-s(n-1)}\frac{1}{2^{-sn}\operatorname{dist}(I,J)}\mathbf{1}_{R_{2s}(I,J)}(z)\\ &= \left|\left\langle (S_{\kappa,\eta})^{-1}f,h_{I;\kappa}\right\rangle \left\langle (S_{\kappa,\eta})^{-1}f,h_{J;\kappa}\right\rangle\right| \frac{2^{s}}{\operatorname{dist}(I,J)}\mathbf{1}_{R_{2s}(I,J)}(z)\\ &= 2^{s+t}\left|\left\langle (S_{\kappa,\eta})^{-1}f,h_{I;\kappa}\right\rangle \left\langle (S_{\kappa,\eta})^{-1}f,h_{J;\kappa}\right\rangle\right| \mathbf{1}_{R_{2s}(I,J)}(z).\end{aligned}$$

Thus we have

$$\begin{aligned}\sum_{t=0}^{s}\Psi_t &\lesssim \sum_{t=0}^{s}\sum_{I,J\in\mathcal{G}_s[S]:\ \operatorname{dist}(I,J)\approx 2^{-t}}\int_{\mathbb{R}^n}\left|f_{\Phi,2s}^{I}*f_{\Phi,2s}^{J}(z)\right|^2dz\\ &\lesssim \sum_{t=0}^{s}\sum_{I,J\in\mathcal{G}_s[S]:\ \operatorname{dist}(I,J)\approx 2^{-t}}\int_{\mathbb{R}^n}\left|2^{s+t}\left|\left\langle (S_{\kappa,\eta})^{-1}f,h_{I;\kappa}\right\rangle \left\langle (S_{\kappa,\eta})^{-1}f,h_{J;\kappa}\right\rangle\right|\mathbf{1}_{R_{2s}(I,J)}(z)\right|^2dz\\ &\lesssim \sum_{t=0}^{s}\sum_{I,J\in\mathcal{G}_s[S]:\ \operatorname{dist}(I,J)\approx 2^{-t}}2^{2s+2t}\left|\left\langle (S_{\kappa,\eta})^{-1}f,h_{I;\kappa}\right\rangle \left\langle (S_{\kappa,\eta})^{-1}f,h_{J;\kappa}\right\rangle\right|^2\left|R_{2s}(I,J)\right|\\ &\lesssim \sum_{t=0}^{s}\sum_{I,J\in\mathcal{G}_s[S]:\ \operatorname{dist}(I,J)\approx 2^{-t}}2^{-s(n-2)}2^{t}\left|\left\langle (S_{\kappa,\eta})^{-1}f,h_{I;\kappa}\right\rangle \left\langle (S_{\kappa,\eta})^{-1}f,h_{J;\kappa}\right\rangle\right|^2\equiv\sum_{t=0}^{s}\Omega_t,\end{aligned}$$

where we have defined Ω_t to be the bound for Ψ_t obtained above.

Now recall that

$$\left\|(\mathsf{Q}_U^s)^{\spadesuit}f\right\|_{L^4(\lambda_{n-1})}^4\approx 2^{s(n-1)}\sum_{I\in\mathcal{G}_s[U]}\left\langle (S_{\kappa,\eta})^{-1}f,h_{I;\kappa}^{n-1}\right\rangle^4.$$

Thus for $0<t<s$ we have

$$\begin{aligned}\Omega_t &\lesssim \sum_{I,J\in\mathcal{G}_s[U]:\ \operatorname{dist}(I,J)\approx 2^{-t}}2^{-s(n-2)}2^{t}\left|\left\langle (S_{\kappa,\eta})^{-1}f,h_{I;\kappa}\right\rangle \left\langle (S_{\kappa,\eta})^{-1}f,h_{J;\kappa}\right\rangle\right|^2\\ &\lesssim 2^{-s(n-2)}2^{t}\sum_{I,J\in\mathcal{G}_s[U]:\ \operatorname{dist}(I,J)\approx 2^{-t}}\left|\left\langle (S_{\kappa,\eta})^{-1}f,h_{I;\kappa}\right\rangle\right|^4\\ &\lesssim 2^{-s(n-2)}2^{t}2^{(s-t)(n-1)}\sum_{I\in\mathcal{G}_s[U]}\left|\left\langle (S_{\kappa,\eta})^{-1}f,h_{I;\kappa}\right\rangle\right|^4=2^{-t(n-2)}2^{-s(n-2)}\left\|(\mathsf{Q}_U^s)^{\spadesuit}f\right\|_{L^4(S)}^4,\end{aligned}$$

since

$$\#\left\{J\in\mathcal{G}_s[S]:\operatorname{dist}(I,J)\approx 2^{-t}\right\}\approx\frac{\text{volume of annulus}}{\text{volume of cube}}\approx\frac{2^{-t(n-1)}}{2^{-s(n-1)}},$$

which then gives

$$\sum_{t=0}^{s}\Psi_t\lesssim\sum_{t=0}^{s}\Omega_t\lesssim\sum_{t=0}^{s}2^{-t(n-2)}2^{-s(n-2)}\left\|(\mathsf{Q}_U^s)^{\spadesuit}f\right\|_{L^4(S)}^4\approx 2^{-s(n-2)}\left\|(\mathsf{Q}_U^s)^{\spadesuit}f\right\|_{L^4(S)}^4.$$

Similarly we obtain

$$\Psi\lesssim 2^{-s(n-2)}\left\|(\mathsf{Q}_U^s)^{\spadesuit}f\right\|_{L^4(S)}^4,$$

and adding these results gives

$$\mathbb{E}_{2^{\mathcal{G}}}^{\mu}\left\Vert \Lambda_{\mathcal{A}_{\mathbf{a}}\mathsf{Q}_{U}^{s}}^{2s}f\right\Vert _{L^{4}(\lambda_{n})}^{4}\lesssim 2^{-s(n-2)}\left\Vert (\mathsf{Q}_{U}^{s})^{S_{\kappa,\eta}}f\right\Vert _{L^{4}(\mathbb{R}^{n-1})}^{4}\lesssim 2^{-s(n-2)}\left\Vert f\right\Vert _{L^{4}(\mathbb{R}^{n-1})}^{4},$$

which is the second line in (5.7).

6. Control of the below form

Combining the above principles of decay, and staying the introduction of absolute values until the very end, we will be able to obtain estimates on the inner products $\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle$, which will lead to the following form bounds for some fixed $\delta>0$ depending only on n and p,

$$\left\vert \mathsf{B}_{\mathrm{below}}^{k,d}(f,g)\right\vert \lesssim 2^{-\delta(|d|+|k|)}\left\Vert f\right\Vert _{L^{p}}\left\Vert g\right\Vert _{L^{p'}}\ ,\qquad \text{for } p>\frac{2n}{n-1}.$$

In fact we obtain stronger bounds in which the absolute values are inside the sum. Indeed, if we define

$$\left\vert \mathsf{B}_{\mathrm{below}}\right\vert (f,g)\equiv\sum_{(I,J)\in\mathcal{P}_{0}}\left\vert \left\langle T\bigtriangleup_{I;\kappa}^{n-1,\eta}f,\bigtriangleup_{J;\kappa}^{n,\eta}g\right\rangle \right\vert ,$$

we prove in this section that

$$\left\vert \mathsf{B}_{\mathrm{below}}\right\vert (f,g)\lesssim\left\Vert f\right\Vert _{L^{p}}\left\Vert g\right\Vert _{L^{p'}}\ ,\qquad \text{for } p>\frac{2n}{n-1}. \tag{6.1}$$

We will begin with the two easier cases involving $d\leq0$, since each of these cases requires just one of the decay principles described above.

Later we turn to the subforms involving $d\geq0$, which are harder to control as each of them requires combining two of the decay principles described above.

Remark 36. *The next result shows in particular that the basic form $\mathsf{B}_{\mathrm{below}}^{0,0}(f,g)$ is bounded using only the crude estimate (4.5), and the strict restriction to $p>\frac{2n}{n-1}$. See also the Direct Argument in Subsubsection 9.3.1 for a much shorter proof of essentially the same result.*

6.1. Subforms with $k\geq0,d\leq0$.

Here is the conclusion of this first subsection.

Lemma 37. *Fix $s\in\mathbb{N}$. Then*

$$\sum_{k\geq0}\sum_{d\leq0}\left\vert \mathsf{B}_{\mathrm{below}}^{k,d}(f,g)\right\vert \leq\sum_{k\geq0}\sum_{d\leq0}\sum_{(I,J)\in\mathcal{P}_{0}^{k,d}}\left\vert \left\langle T\bigtriangleup_{I;\kappa}^{n-1,\eta}f,\bigtriangleup_{J;\kappa}^{n,\eta}g\right\rangle \right\vert \lesssim\left\Vert f\right\Vert _{L^{p}}\left\Vert g\right\Vert _{L^{p'}}\ ,\qquad \textit{for } p\geq\frac{2n}{n-1}. \tag{6.2}$$

To prove Lemma 37, we just need the estimate (4.6) that used radial integration by parts, namely,

$$\left\vert \left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle \right\vert \leq C_{N}2^{-kN}\left\Vert h_{I;\kappa}^{n-1,\eta}\right\Vert _{L^{1}}\left\Vert h_{J;\kappa}^{n,\eta}\right\Vert _{L^{1}}\approx2^{-kN}\sqrt{|I|\,|J|},\qquad k\geq0.$$

Let $I_{\eta}\equiv(1+\eta)I$ so that $\mathrm{Supp}\bigtriangleup_{I;\kappa}^{n-1,\eta}f\subset I_{\eta}$. Note also that $|I_{\eta}|\approx|I|$. Then we have from (4.6),

$$\begin{aligned}
\left\vert \mathsf{B}_{\mathrm{below}}^{k,d}(f,g)\right\vert &\leq\sum_{(I,J)\in\mathcal{P}_{0}^{k,d}}\left\vert \left\langle T\bigtriangleup_{I;\kappa}^{n-1,\eta}f,\bigtriangleup_{J;\kappa}^{n,\eta}g\right\rangle \right\vert \leq\sum_{(I,J)\in\mathcal{P}_{0}^{k,d}}2^{-kN}\left(\int_{I_{\eta}}\left\vert \bigtriangleup_{I;\kappa}^{n-1,\eta}f(x)\right\vert dx\right)\left(\int_{J_{\eta}}\left\vert \bigtriangleup_{J;\kappa}^{n,\eta}g(\xi)\right\vert d\xi\right)\\
&=2^{-kN}\int_{\mathbb{R}^{n}}\sum_{(I,J)\in\mathcal{P}_{0}^{k,d}}\left(\int_{I_{\eta}}\left\vert \bigtriangleup_{I;\kappa}^{n-1,\eta}f(x)\right\vert dx\right)\mathbf{1}_{J_{\eta}}(\xi)\left\vert \bigtriangleup_{J;\kappa}^{n,\eta}g(\xi)\right\vert d\xi\\
&\leq2^{-kN}\int_{\mathbb{R}^{n}}\sqrt{\sum_{(I,J)\in\mathcal{P}_{0}^{k,d}}\left(\int_{I_{\eta}}\left\vert \bigtriangleup_{I;\kappa}^{n-1,\eta}f(x)\right\vert dx\mathbf{1}_{J_{\eta}}(\xi)\right)^{2}}\sqrt{\sum_{(I,J)\in\mathcal{P}_{0}^{k,d}}\left\vert \bigtriangleup_{J;\kappa}^{n,\eta}g(\xi)\right\vert ^{2}}d\xi\\
&\lesssim2^{-kN}\left(\int_{\mathbb{R}^{n}}\left(\sum_{(I,J)\in\mathcal{P}_{0}^{k,d}}\left(\int_{I}\left\vert \bigtriangleup_{I;\kappa}^{n-1,\eta}f(x)\right\vert dx\mathbf{1}_{J_{\eta}}(\xi)\right)^{2}\right)^{\frac{p}{2}}d\xi\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^{n}}\left(\sum_{(I,J)\in\mathcal{P}_{0}^{k,d}}\left\vert \bigtriangleup_{J;\kappa}^{n,\eta}g(\xi)\right\vert ^{2}\right)^{\frac{p'}{2}}d\xi\right)^{\frac{1}{p'}}\\
&\equiv2^{-kN}\Gamma_{1}\Gamma_{2}\ ,
\end{aligned}$$

where

$$\Gamma_2^{p'}=\int_{\mathbb{R}^n}\left(\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left|\triangle_{J;\kappa}^{n,\eta}g(\xi)\right|^2\right)^{\frac{p'}{2}}d\xi=\int_{\mathbb{R}^n}\left(\sum_{J\in\mathcal{D}}\left(\sum_{I\in\mathcal{G}:\ (I,J)\in\mathcal{P}_0^{k,d}}1\right)\left|\triangle_{J;\kappa}^{n,\eta}g(\xi)\right|^2\right)^{\frac{p'}{2}}d\xi.$$

We now choose a dyadic cube $I_J\in\mathcal{G}$ that approximates the spherical projection $\pi_{\tan}(J)$ of J. So fix $J\in\mathcal{D}$ and let $I_J\in\mathcal{G}$ satisfy

$$c_n\ell(\pi_{\tan}(J))\leq\ell(I_J)\leq\ell(\pi_{\tan}(J))\ \text{ and }I_J\subset\pi_{\tan}(J),$$

where $\pi_{\tan}(J)$ is the spherical projection J onto $\mathbb{S}^{n-1}$, and where $c_n>0$ is chosen small enough that such a cube I_J exists.

Now $(I,J)\in\mathcal{P}_0^{k,d}$ if and only if

$$\pi_{\tan}J\subset\Phi(C_{\text{pseudo}}I)\ \text{ and }\frac{2^{d-1}}{\ell(I)^2}\leq\operatorname{dist}(0,J)\leq\frac{2^{d+1}}{\ell(I)^2},$$

which is essentially equivalent to

$$I\supset\pi_{\tan}J\supset I_J\ \text{ and }\sqrt{\frac{2^{d-1}}{2\operatorname{dist}(0,J)}}\leq\ell(I)\leq\sqrt{\frac{2^{d+1}}{\operatorname{dist}(0,J)}}.$$

Thus for fixed $J\in\mathcal{D}_k$ where

$$\mathcal{D}_k\equiv\left\{J\in\mathcal{D}:\ell(J)=2^k\right\},$$

the set of cubes $I\in\mathcal{G}$ with $(I,J)\in\mathcal{P}_0^{k,d}$ is contained in the finite tower of dyadic cubes $\left\{\pi^{(k)}I_J\right\}_{k=d-A}^{d+A}$ for some fixed $A\in\mathbb{N}$. It follows that $\sum_{I\in\mathcal{G}:\ (I,J)\in\mathcal{P}_0^{k,d}}1\leq 2A$ and so

$$\Gamma_2^{p'}=\int_{\mathbb{R}^n}\left(\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left|\triangle_{J;\kappa}^{n,\eta}g(\xi)\right|^2\right)^{\frac{p'}{2}}d\xi\leq\int_{\mathbb{R}^n}\left(\sum_{J\in\mathcal{D}_k}2A\left|\triangle_{J;\kappa}^{n,\eta}g(\xi)\right|^2\right)^{\frac{p'}{2}}d\xi\lesssim\|g\|_{L^{p'}}^{p'}\ ,\tag{6.3}$$

by the square function estimate (1.17).

We turn now to estimating Γ_1. Since the cubes J_η in $\mathcal{D}_k$ have bounded overlap with measure roughly 2^{kn},

$$\begin{aligned}\Gamma_1^p&=\int_{\mathbb{R}^n}\left(\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left(\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|dx\mathbf{1}_{J_\eta}(\xi)\right)^2\right)^{\frac{p}{2}}d\xi\\&=\int_{\mathbb{R}^n}\left(\sum_{J\in\mathcal{D}_k}\left\{\sum_{I\in\mathcal{G}[S]:\ (I,J)\in\mathcal{P}_0^{k,d}}\left(\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)^2\right\}\mathbf{1}_{J_\eta}(\xi)\right)^{\frac{p}{2}}d\xi\\&\approx\int_{\mathbb{R}^n}\sum_{J\in\mathcal{D}_k}\left\{\sum_{I\in\mathcal{G}[S]:\ (I,J)\in\mathcal{P}_0^{k,d}}\left(\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)^2\right\}^{\frac{p}{2}}\mathbf{1}_{J_\eta}(\xi)\,d\xi\\&\approx 2^{kn}\sum_{J\in\mathcal{D}_k}\left(\sum_{I\in\mathcal{G}[S]:\ (I,J)\in\mathcal{P}_0^{k,d}}\left(\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)^2\right)^{\frac{p}{2}}.\end{aligned}\tag{6.4}$$

Now for each fixed $J\in\mathcal{D}_k$ and $I\in\mathcal{G}[S]$ with $(I,J)\in\mathcal{P}_0^{k,d}$, we have

$$\begin{aligned}\ell(J)&=2^k,\quad\ell(I)^2\operatorname{dist}(0,J)\approx 2^d,\quad\pi_{\tan}J\subset\Phi(C_{\text{pseudo}}I),\\\ell(I_J)&\approx\ell(\pi_{\tan}J)\approx\frac{\ell(J)}{\operatorname{dist}(0,J)}=\frac{2^k}{\operatorname{dist}(0,J)},\end{aligned}$$

which implies

$$\begin{aligned} \ell(I) &\approx \sqrt{\frac{2^d}{\operatorname{dist}(0,J)}} \approx \sqrt{\frac{2^d \ell(\pi_{\tan} J)}{2^k}} = 2^{\frac{d-k}{2}} \sqrt{\ell(I_J)}, \\ \log_2 \frac{\ell(I)}{\ell(I_J)} &\approx \log_2 \frac{2^{\frac{d-k}{2}}}{\sqrt{\ell(I_J)}} \approx \frac{1}{2}\left(d-k-\log_2 \frac{1}{\ell(I_J)}\right). \end{aligned}$$

Thus with $d^* \equiv \frac{1}{2}\left(d-k-\log_2 \frac{1}{\ell(I_J)}\right)$ and A as in (6.3) above, we have for each $J \in \mathcal{D}$,

$$\begin{aligned} &\left(\sum_{I \in \mathcal{G}[S]:\ (I,J) \in \mathcal{P}_0^{k,d}} \left(\int_{I_\eta} \left|\triangle_{I;\kappa}^{n-1,\eta} f(x)\right| dx\right)^2\right)^{\frac{p}{2}} \leq \left(\sum_{s=d^*-A}^{d^*+A} \left(\int_{\pi^{(m)}(I_J)_\eta} \left|\triangle_{\pi^{(s)}(I_J);\kappa}^{n-1,\eta} f(x)\right| dx\right)^2\right)^{\frac{p}{2}} \\ \leq\ & (2A)^{\frac{p}{2}-1} \sum_{I \in \mathcal{G}[S]:\ (I,J) \in \mathcal{P}_0^{k,d}} \left(\int_{I_\eta} \left|\triangle_{I;\kappa}^{n-1,\eta} f(x)\right| dx\right)^p \approx \sum_{I \in \mathcal{G}[S]:\ (I,J) \in \mathcal{P}_0^{k,d}} \left(\int_{I_\eta} \left|\triangle_{I;\kappa}^{n-1,\eta} f(x)\right| dx\right)^p. \end{aligned}$$

Altogether then,

$$\begin{aligned} \Gamma_1^p &\lesssim 2^{kn} \sum_{J \in \mathcal{D}_k} \sum_{I \in \mathcal{G}[S]:\ (I,J) \in \mathcal{P}_0^{k,d}} \left(\int_{I_\eta} \left|\triangle_{I;\kappa}^{n-1,\eta} f(x)\right| dx\right)^p \\ &\leq 2^{kn} \sum_{J \in \mathcal{D}_k} \sum_{I \in \mathcal{G}[S]:\ (I,J) \in \mathcal{P}_0^{k,d}} |I|^{\frac{p}{2}} \left(\int_{I_\eta} \left|\triangle_{I;\kappa}^{n-1,\eta} f(x)\right|^2 dx\right)^{\frac{p}{2}} \\ &\approx 2^{kn} \sum_{I \in \mathcal{G}[S]} \left(\sum_{J \in \mathcal{D}_k:\ (I,J) \in \mathcal{P}_0^{k,d}} 1\right) |I|^p \left(\frac{1}{|I_\eta|} \int_{I_\eta} \left|\triangle_{I;\kappa}^{n-1,\eta} f(x)\right|^2 dx\right)^{\frac{p}{2}}. \end{aligned} \tag{6.5}$$

Now recall that $\mathcal{P}_0 \equiv \{(I,J) \in \mathcal{G}[S] \times \mathcal{D} : \pi_{\tan}(J) \subset \Phi(C_{\text{pseudo}} I)\}$, and define

$$\mathcal{K}(I) \equiv \bigcup \{J \in \mathcal{D} : \pi_{\tan}(J) \subset \Phi(C_{\text{pseudo}} I)\}.$$

Now for fixed $I \in \mathcal{G}[S]$,

$$\begin{aligned} & \#\left\{J \in \mathcal{D}_k :\ (I,J) \in \mathcal{P}_0^{k,d}\right\} \\ \approx\ & 2^{-kn} |\mathcal{K}_d(I)| \approx 2^{-kn} \left(\frac{2^d}{\ell(I)^2} \ell(I)\right)^{n-1} \frac{2^d}{\ell(I)^2} \\ =\ & 2^{-kn} \frac{2^{dn}}{\ell(I)^{n+1}} = 2^{-kn} 2^{dn} \left(\frac{1}{|I|}\right)^{\frac{n+1}{n-1}}; \\ & \text{where } \mathcal{K}_d(I) \equiv \bigcup \left\{J \subset \mathcal{K}(I) : \frac{2^{d-1}}{\ell(I)^2} \leq \operatorname{dist}(0,J) \leq \frac{2^{d+1}}{\ell(I)^2}\right\}, \end{aligned} \tag{6.6}$$

and so we have

$$\begin{aligned}\Gamma_1^p &\lesssim 2^{kn}\sum_{I\in\mathcal{G}[U]}\left(\#\left\{J\in\mathcal{D}_k:\ (I,J)\in\mathcal{P}_0^{k,d}\right\}\right)|I|^p\left(\frac{1}{|I_\eta|}\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f\right|^2\right)^{\frac{p}{2}}\\ &\lesssim 2^{kn}2^{-kn}2^{dn}\sum_{I\in\mathcal{G}[U]}|I|^{p-\frac{n+1}{n-1}}\left(\frac{1}{|I_\eta|}\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f\right|^2\right)^{\frac{p}{2}}\\ &= 2^{dn}\int_S\sum_{I\in\mathcal{G}[U]}|I|^{p-\frac{n+1}{n-1}-1}\left(\frac{1}{|I_\eta|}\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f\right|^2\right)^{\frac{p}{2}}\mathbf{1}_I(x)\,dx\\ &\leq 2^{dn}\int_S\sum_{I\in\mathcal{G}[U]}\left(\frac{1}{|I_\eta|}\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f\right|^2\mathbf{1}_I(x)\right)^{\frac{p}{2}}dx\ ,\end{aligned}$$

if $p\geq\frac{2n}{n-1}$. Now using Hölder's inequality with $\frac{p}{2}>1$, and the Fefferman Stein vector valued maximal inequality,we can continue with

$$\begin{aligned}\Gamma_1^p &\lesssim 2^{dn}\int_S\left(\sum_{I\in\mathcal{G}[U]}\frac{1}{|I_\eta|}\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f\right|^2\mathbf{1}_I(x)\right)^{\frac{p}{2}}dx\lesssim 2^{dn}\int_S\left(\sum_{I\in\mathcal{G}[U]}\left(M\left|\triangle_{I;\kappa}^{n-1,\eta}f\right|^2\right)(x)\right)^{\frac{p}{2}}dx\\ &\lesssim 2^{dn}\int_S\left(\sum_{I\in\mathcal{G}[U]}\left|\triangle_{I;\kappa}^{n-1,\eta}f\right|^2(x)\right)^{\frac{p}{2}}dx\lesssim 2^{dn}\|f\|_{L^p}^p\ ,\end{aligned}\tag{6.7}$$

by the square function estimate (1.17). Thus we have proved,

$$\left|\mathsf{B}_{\text{below}}^{k,d}(f,g)\right|\lesssim 2^{-kN}2^{\frac{dn}{p}}\|f\|_{L^p}\|g\|_{L^{p'}}\ ,\qquad\text{for }k\geq 0\text{ and }d\leq 0,$$

which gives

$$\sum_{k\geq 0}\sum_{d\leq 0}\left|\mathsf{B}_{\text{below}}^{k,d}(f,g)\right|\lesssim\|f\|_{L^p}\|g\|_{L^{p'}}\ ,\qquad\text{for }p\geq\frac{2n}{n-1}.$$

6.2. **Subforms with** $k\leq 0,d\leq 0$**.** This case also requires just one principle of decay, but this time we use the moment vanishing decay principle instead of the radial integration by parts decay principle. From (4.13) we have

$$\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle=\int_S e^{-i\Phi(x)\cdot c_J}h_{I;\kappa}^{n-1,\eta}(x)\left\{\int_{\mathbb{R}^n}R_\kappa\left(-i\Phi(x)\cdot(\xi-c_J)\right)h_{J;\kappa}^{n,\eta}(\xi)\,d\xi\right\}dx,$$

and then from (4.14), we obtain the estimate,

$$\begin{aligned}\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right| &\leq \int_S\left|h_{I;\kappa}^{n-1,\eta}(x)\right|\int_{\mathbb{R}^n}\frac{|\Phi(x)\cdot(\xi-c_J)|^\kappa}{(\kappa+1)!}\left|h_{J;\kappa}^{n,\eta}(\xi)\right|d\xi dx\\ &\lesssim \ell(J)^\kappa\left\|\varphi_I^\eta\right\|_{L^1}\left\|\psi_J^\eta\right\|_{L^1}\approx 2^{-|k|\kappa}\sqrt{|I|\,|J|}.\end{aligned}$$

The proof is now virtually the same as that in the previous subsection, but using the above estimate instead, and results in the bound,

$$\left|\mathsf{B}_{\text{below}}^{k,d}(f,g)\right|\lesssim 2^{-|k|\kappa}2^{\frac{dn}{p}}\|f\|_{L^p}\|g\|_{L^{p'}}\ ,\qquad\text{for }k\leq 0\text{ and }d\leq 0,$$

which gives

$$\sum_{k\leq 0}\sum_{d\leq 0}\left|\mathsf{B}_{\text{below}}^{k,d}(f,g)\right|\lesssim\|f\|_{L^p}\|g\|_{L^{p'}}\ ,\qquad\text{for }p\geq\frac{2n}{n-1}.$$

6.3. **Subforms with $k \leq 0, d \geq 0$.** Here we will use the vanishing moments of $h_{J;\kappa}^{n,\eta}$ together with stationary phase. In the case $k \leq 0$ and $d \geq 0$, we have from (4.13), which used the vanishing moments of $h_{J;\kappa}^{n,\eta}$,

$$\left\langle Th_{I;\kappa}^{n-1,\eta}, h_{J;\kappa}^{n,\eta} \right\rangle = \int_S e^{-i\Phi(x)\cdot c_J} h_{I;\kappa}^{n-1,\eta}(x) \left\{ \int_{\mathbb{R}^n} R_\kappa \left(-i\Phi(x) \cdot (\xi - c_J) \right) h_{J;\kappa}^{n,\eta}(\xi) \, d\xi \right\} dx,$$

and using the change of variable $\xi \to (y, \lambda)$ in (3.7) with $\frac{c_J}{|c_J|} = \Phi(y_J)$, this can be written,

$$\begin{aligned} (6.8) \qquad & \left\langle Th_{I;\kappa}^{n-1,\eta}, h_{J;\kappa}^{n,\eta} \right\rangle \\ = & \int_{\mathbb{R}^n} \left\{ \int_S e^{-i\lambda\phi(x,y_J)} h_{I;\kappa}^{n-1,\eta}(x) R_\kappa \left(-i\lambda\Phi(x) \cdot \left(\Phi(y) - \frac{|c_J|}{\lambda} \Phi(y_J) \right) \right) dx \right\} h_{J;\kappa}^{n,\eta}(\lambda\Phi(y)) \frac{dy}{\sqrt{1-|y|^2}} \lambda^{n-1} d\lambda \\ = & \int_{\mathbb{R}^n} \mathcal{I}_{\widehat{\varphi_I^\eta},\phi}(y_J, \lambda) h_{J;\kappa}^{n,\eta}(\lambda\Phi(y)) \frac{dy}{\sqrt{1-|y|^2}} \lambda^{n-1} d\lambda, \end{aligned}$$

where

$$\mathcal{I}_{\widehat{\varphi_I^\eta},\phi}(y_J, \lambda) = \int_S e^{-i\lambda\phi(x,y_J)} \widehat{\varphi_I^\eta}(x, y, y_J) \, dx,$$

and

$$\widehat{\varphi_I^\eta}(x, y_J, y) \equiv h_{I;\kappa}^{n-1,\eta}(x) R_\kappa \left(-i\lambda\Phi(x) \cdot \left(\Phi(y) - \frac{|c_J|}{\lambda} \Phi(y_J) \right) \right) = h_{I;\kappa}^{n-1,\eta}(x) R_\kappa \left(-i\Phi(x) \cdot (\xi - c_J) \right),$$

where R_κ satisfies the estimates,

$$\begin{aligned} (6.9) \qquad |R_\kappa(ib)| &= \left| \int_0^1 e^{itb} (ib)^\kappa \frac{(1-t)^\kappa}{\kappa!} dt \right| \lesssim \frac{|b|^\kappa}{\kappa!}, \\ \left| R_\kappa^{(\ell)}(b) \right| &= \left| \int_0^1 \partial_b^\ell \left[e^{itb} (ib)^\kappa \right] \frac{(1-t)^\kappa}{\kappa!} dt \right| \lesssim \max \left\{ |b|^{\kappa-\ell}, |b|^\kappa \right\}, \end{aligned}$$

and y_J is the unique point in S such that $\frac{c_J}{|c_J|} = \Phi(y_J)$.

Theorem 28 with $M = 0$ gives the asymptotic expansion,

$$\mathcal{I}_{\widehat{\varphi_I^\eta},\phi}(y_J, \lambda) = \mathfrak{P}_{\widehat{\varphi_I^\eta},\phi}(y_J, \lambda) + \mathfrak{R}^{(1)}_{\widehat{\varphi_I^\eta},\phi}(y_J, \lambda),$$

where

$$\mathfrak{P}_{\widehat{\varphi_I^\eta},\phi}(y_J, \lambda) = \left(\frac{2\pi}{\lambda} \right)^{\frac{n}{2}} \frac{e^{i \operatorname{sgn}\left[\partial_x^2 \phi(X(y_J), y_J)\right] \frac{\pi}{4} + \lambda\phi(X(y_J), y_J)}}{\sqrt{\left| \partial_x^2 \phi(X(y_J), y_J) \right|}} \widehat{\varphi_I^\eta}(X(y_J), y_J, y),$$

and

$$\begin{aligned} \mathfrak{R}^{(1)}_{\widehat{\varphi_I^\eta},\phi}(y_J, \lambda) &= \left(\frac{2\pi}{\lambda} \right)^{\frac{n}{2}} \frac{e^{i\left[\operatorname{sgn} B(y_J) \frac{\pi}{4} + \lambda\phi(X(y_J), y_J)\right]}}{\sqrt{\det B(y_J)}} \\ & \quad \times \int \mathcal{F}_z^{-1} \left(\left[\frac{\left\langle i\partial_z, B(y_J)^{-1} \partial_z \right\rangle}{2\lambda} \right]^1 f \right)(\zeta) R_1 \left(-i \frac{\zeta^{\mathrm{tr}} B(y_J)^{-1} \zeta}{2\lambda} \right) d\zeta, \end{aligned}$$

and where

$$R_1(ib) = \int_0^1 e^{itb} (ib)^1 \frac{(1-t)^1}{(M+1)!} dt, \qquad \text{for } b \in \mathbb{R},$$

and

$$(6.10) \qquad f(z, y_J, y) \equiv \frac{\widehat{\varphi_I^\eta}\left(\Psi_y^{-1}(z), y_J, y \right)}{\det\left[(\partial_x \Psi) \left(\Psi_y^{-1}(z) \right) \right]}.$$

We can rewrite the principal term as

$$\begin{aligned}\mathfrak{P}_{\widehat{\varphi_I^\eta},\phi}(y_J,\lambda) &= \left(\frac{2\pi}{\lambda}\right)^{\frac{n-1}{2}} \frac{e^{i\,\mathrm{sgn}\left[\partial_x^2\phi(X(y_J),y_J)\right]\frac{\pi}{4}+\lambda\phi(X(y_J),y_J)}}{\sqrt{|\det B(y_J)|}}\widehat{\varphi_I^\eta}(X(y),y_J,y)\\ &= \left(\frac{2\pi}{\lambda}\right)^{\frac{n-1}{2}} e^{i\frac{(n-1)\pi}{4}+\lambda\sqrt{1-|y_J|^2}}\widehat{\varphi_I^\eta}(y_J,y_J,y)\\ &= e^{-\frac{(n-1)\pi}{4}}e^{i|\xi|}\left(\frac{2\pi}{|\xi|}\right)^{\frac{n-1}{2}}\frac{\xi_n}{|\xi|}\widehat{\varphi_I^\eta}\left(\frac{c_J'}{|c_J|},\frac{c_J'}{|c_J|},\frac{\xi'}{|\xi|}\right),\end{aligned}$$

and the remainder term as

$$\begin{aligned}\mathfrak{R}^{(1)}_{\widehat{\varphi_I^\eta},\phi}(y_J,\lambda) &= \left(\frac{2\pi}{\lambda}\right)^{\frac{n-1}{2}}\frac{e^{i\left[\mathrm{sgn}\,B(y_J)\frac{\pi}{4}+\lambda\phi(X(y_J),y_J)\right]}}{\sqrt{|\det B(y)|}}\\ &\quad\times\int\mathcal{F}_z^{-1}\left(\left[\frac{\left\langle i\partial_z,B(y_J)^{-1}\partial_z\right\rangle}{2\lambda}\right]f\right)(\zeta)\,R_1\left(-i\frac{\zeta^{\mathrm{tr}}B(y_J)^{-1}\zeta}{2\lambda}\right)d\zeta.\end{aligned}\tag{6.11}$$

Now we compute that for $x\in I$ and $y\in\pi_{\tan}J$,

$$\begin{aligned}\left|\lambda\,\Phi(x)\cdot\left(\Phi(y)-\frac{|c_J|}{\lambda}\Phi(y_J)\right)\right| &\lesssim \lambda\left|\Phi(y)-\frac{|c_J|}{\lambda}\Phi(y_J)\right|\lesssim\ell(J),\\ \text{and }\left|\lambda\,\partial_x^N\Phi(x)\cdot\left(\Phi(y)-\frac{|c_J|}{\lambda}\Phi(y_J)\right)\right| &\lesssim \lambda\left|\Phi(y)-\frac{|c_J|}{\lambda}\Phi(y_J)\right|\lesssim\ell(J),\qquad\text{for } N\geq 1.\end{aligned}\tag{6.12}$$

Since $\left|\lambda\Phi(x)\cdot\left(\Phi(y)-\frac{|c_J|}{\lambda}\Phi(y_J)\right)\right|\lesssim\ell(J)\lesssim 1$, the modulus of the inner product $\left\langle\mathfrak{P}_{\widehat{\varphi_I^\eta},\phi},h^{n,\eta}_{J;\kappa}\right\rangle$ is thus bounded by,

$$\begin{aligned}&\left|\left\langle\mathfrak{P}_{\widehat{\varphi_I^\eta},\phi},h^{n,\eta}_{J;\kappa}\right\rangle\right|\leq\int_{\mathbb{R}^n}\left|\mathfrak{P}_{\widehat{\varphi_I^\eta},\phi}(\xi)\,h^{n,\eta}_{J;\kappa}(\xi)\right|d\xi\leq\left\|\mathfrak{P}_{\widehat{\varphi_I^\eta},\phi}\right\|_{L^\infty}\left\|h^{n,\eta}_{J;\kappa}\right\|_{L^\infty}|J|\\ \lesssim&\left(\frac{1}{\mathrm{dist}\,0,J}\right)^{\frac{n-1}{2}}\left\|\widehat{\varphi_I^\eta}\right\|_{L^\infty}\sqrt{|J|}\lesssim\left(\frac{1}{\mathrm{dist}\,0,J}\right)^{\frac{n-1}{2}}\frac{1}{\sqrt{|I|}}\sup_{y\in\pi_{\tan}J}\left|R_\kappa\left(-i\lambda\Phi(x)\cdot\left(\Phi(y)-\frac{|c_J|}{\lambda}\Phi(y_J)\right)\right)\right|\sqrt{|J|}\\ \lesssim&\left(\frac{1}{\mathrm{dist}(0,J)}\right)^{\frac{n-1}{2}}\frac{1}{\sqrt{|I|}}\ell(J)^\kappa\sqrt{|J|}=\left(\frac{1}{\ell(I)^2\,\mathrm{dist}(0,J)}\right)^{\frac{n-1}{2}}\ell(J)^\kappa\sqrt{|I|\,|J|}\\ =&\left(\frac{1}{\ell(I)^2\,\mathrm{dist}(0,J)}\right)^{\frac{n-1}{2}}\ell(J)^\kappa\sqrt{|I|\,|J|}\approx 2^{-d\frac{n-1}{2}}2^{-|k|\kappa}\sqrt{|I|\,|J|}\\ \lesssim&\ 2^{-d\frac{n-1}{2}}2^{-|k|\kappa}\sqrt{|I|\,|J|}.\end{aligned}$$

To estimate the remainder term (6.11), we thank Cristian Rios for the following argument, which corrects and simplifies an earlier one in a previous version of this paper. We first need to estimate derivatives of f in (6.10). From the identity

$$\frac{\partial}{\partial x^\alpha}R_\kappa(ib(x))=\sum_{0\neq\beta\leq\alpha}\binom{\alpha}{\beta}\frac{d^{|\beta|}R_\kappa}{db^{|\beta|}}(ib)\,\frac{\partial}{\partial x^{\alpha-\beta}}\prod_{\ell=1}^{n-1}\left(\partial_{x_\ell}(ib(x))\right)^{\beta_\ell}\tag{6.13}$$

With $R=R_\kappa$, $b(x)=-\lambda\Phi(x)\cdot\left(\Phi(y)-\frac{|c_J|}{\lambda}\Phi(y_J)\right)$, by (6.9) and the fact that $\left|\frac{\partial}{\partial x^\beta}ib(x)\right|\lesssim\ell(J)$, we have that

$$\left|\frac{\partial}{\partial x^\alpha}R_\kappa\left(-i\lambda\Phi(x)\cdot\left(\Phi(y)-\frac{|c_J|}{\lambda}\Phi(y_J)\right)\right)\right|\lesssim\sum_{j=1}^{|\alpha|}\left|\frac{d^{|\beta|}R_\kappa}{db^{|\beta|}}(ib)\right|\ell(J)^{|\beta|}\lesssim\ell(J)^\kappa.$$

Then, whever $\kappa \geq |\alpha|$ we have

$$\begin{aligned} \left|\partial_{x^\alpha} \widehat{\varphi_I^\eta}(x)\right| &\leq \left|\partial_{x^\alpha} \widehat{\varphi_I^\eta}(x, y_J, y)\right| \\ &= \left|\sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \left(\frac{\partial}{\partial x^{\beta-\alpha}} h_{I;\kappa}^{n-1,\eta}(x)\right) \left(\frac{\partial}{\partial x^{\beta-\alpha}} R_\kappa(ib(x))\right)\right| \\ &\lesssim \sum_{j=0}^{|\alpha|} \frac{\mathbf{1}_{I_\eta}(x)}{\sqrt{|I|}} \frac{\ell(J)^\kappa}{\ell(I)^j} \lesssim \frac{\mathbf{1}_{I_\eta}(x)}{\sqrt{|I|}} \frac{\ell(J)^\kappa}{\ell(I)^{|\alpha|}}. \end{aligned} \tag{6.14}$$

Now we estimate the first factor in the integral in (6.11)

$$\mathcal{F}_z^{-1}\left(\left[\frac{\left\langle i\partial_z, B(y_J)^{-1}\partial_z\right\rangle}{2\lambda}\right] f\right)(\zeta) = \int_{\Psi(I_\eta)} \left(\left[\frac{\left\langle i\partial_z, B(y_J)^{-1}\partial_z\right\rangle}{2\lambda}\right] \frac{\widehat{\varphi_I^\eta}\left(\Psi_y^{-1}(z), y_J, y\right)}{\det\left[(\partial_x \Psi)\left(\Psi_y^{-1}(z)\right)\right]}\right) e^{iz\cdot\zeta}\, dz$$

Since Ψ_y is a diffeomorphism we have that $\left|\det\left[(\partial_x\Psi)\left(\Psi_y^{-1}(z)\right)\right]\right| \approx 1$, and $\left|\partial_z^j \det\left[(\partial_x\Psi)\left(\Psi_y^{-1}(z)\right)\right]\right| \lesssim C_j$ for $j \geq 1$. Then by the worst case $|\alpha| = 2$ in (6.14) we obtain

$$\left|\left[\frac{\left\langle i\partial_z, B(y_J)^{-1}\partial_z\right\rangle}{2\lambda}\right] \frac{\widehat{\varphi_I^\eta}\left(\Psi_y^{-1}(z), y_J, y\right)}{\det\left[(\partial_x\Psi)\left(\Psi_y^{-1}(z)\right)\right]}\right| \lesssim \frac{1}{\lambda}\left|\partial_x^2 \widehat{\varphi_I^\eta}(x, y_J, y)\right| \lesssim \frac{1}{\lambda} \frac{\mathbf{1}_{I_\eta}(x)}{\sqrt{|I|}} \frac{\ell(J)^\kappa}{\ell(I)^2}.$$

Hence,

$$\left|\mathcal{F}_z^{-1}\left(\left[\frac{\left\langle i\partial_z, B(y_J)^{-1}\partial_z\right\rangle}{2\lambda}\right] f\right)(\zeta)\right| \lesssim \int_{\Psi(I_\eta)} \frac{1}{\lambda} \frac{\mathbf{1}_{I_\eta}(x)}{\sqrt{|I|}} \frac{\ell(J)^\kappa}{\ell(I)^2}\, d\zeta \lesssim \frac{1}{\lambda} \frac{\ell(J)^\kappa}{\ell(I)^2}\sqrt{|I|}. \tag{6.15}$$

From the identity $e^{iz\cdot\zeta} = |\zeta|^{-2N}\left(-i\sum_{j=1}^{n-1}\zeta_j\partial_{z_j}\right)^N e^{iz\cdot\zeta}$, we can also write

$$\left|\mathcal{F}_z^{-1}\left(\left[\frac{\left\langle i\partial_z, B(y_J)^{-1}\partial_z\right\rangle}{2\lambda}\right] f\right)(\zeta)\right| = |\zeta|^{-2N}\left|\int_{\Psi(I_\eta)} \left(\left(i\sum_{j=1}^{n-1}\zeta_j\partial_{z_j}\right)^{Ne^{iz\cdot\zeta}} \left[\frac{\left\langle i\partial_z, B(y_J)^{-1}\partial_z\right\rangle}{2\lambda}\right] f\right) e^{iz\cdot\zeta}\, dz\right|$$

and since, as before, we have the bounds

$$\left|\left(i\sum_{j=1}^{n-1}\zeta_j\partial_{z_j}\right)^N \left[\frac{\left\langle i\partial_z, B(y_J)^{-1}\partial_z\right\rangle}{2\lambda}\right] f\right| \lesssim \frac{|\zeta|^N}{\lambda}\left|\partial_x^{N+2}\widehat{\varphi_I^\eta}(x, y_J, y)\right| \lesssim \frac{|\zeta|^N}{\lambda} \frac{\mathbf{1}_{I_\eta}(x)}{\sqrt{|I|}} \frac{\ell(J)^\kappa}{\ell(I)^{N+2}},$$

hence,

$$\left|\mathcal{F}_z^{-1}\left(\left[\frac{\left\langle i\partial_z, B(y_J)^{-1}\partial_z\right\rangle}{2\lambda}\right] f\right)(\zeta)\right| \lesssim |\zeta|^{-2N}\int_{\Psi(I_\eta)} \frac{|\zeta|^N}{\lambda} \frac{\mathbf{1}_{I_\eta}(x)}{\sqrt{|I|}} \frac{\ell(J)^\kappa}{\ell(I)^{N+2}}\, dz \lesssim \frac{1}{\lambda}\frac{1}{|\zeta|^N} \frac{\ell(J)^\kappa}{\ell(I)^{N+2}}\sqrt{|I|}.$$

Combining this with (6.15) yields

$$\left|\mathcal{F}_z^{-1}\left(\left[\frac{\left\langle i\partial_z, B(y_J)^{-1}\partial_z\right\rangle}{2\lambda}\right] f\right)(\zeta)\right| \lesssim \frac{1}{\lambda}\sqrt{|I|}\frac{\ell(J)^\kappa}{\ell(I)^2}\min\left\{1, \frac{1}{|\zeta|^N}\frac{1}{\ell(I)^N}\right\}.$$

Then, from (6.11) and the fact that $|R_1(ib)| \leq |b|$, we obtain

$$\begin{aligned}
(6.16) \qquad & \left|\mathfrak{R}^{(1)}_{\widehat{\varphi^\eta_I},\phi}(y_J,\lambda)\right| \\
= & \left|\frac{\left(\frac{2\pi}{\lambda}\right)^{\frac{n-1}{2}}}{\sqrt{|\det B(y)|}}\right|\left|\int \mathcal{F}_z^{-1}\left(\left[\frac{\left\langle i\partial_z, B(y_J)^{-1}\partial_z\right\rangle}{2\lambda}\right] f\right)(\zeta)\, R_1\left(-i\frac{\zeta^{\mathrm{tr}}B(y_J)^{-1}\zeta}{2\lambda}\right) d\zeta\right| \\
\lesssim & \frac{1}{\lambda^{\frac{n-1}{2}}}\int_{\mathbb{R}^{n-1}}\frac{1}{\lambda}\sqrt{|I|}\frac{\ell(J)^\kappa}{\ell(I)^2}\min\left\{1,\frac{1}{|\zeta|^N}\frac{1}{\ell(I)^N}\right\}\frac{|\zeta|^2}{\lambda}\, d\zeta \\
= & \frac{1}{\lambda^{\frac{n-1}{2}+2}}\frac{\ell(J)^\kappa}{\ell(I)^2}\sqrt{|I|}\int_{\mathbb{R}^{n-1}}\min\left\{1,\frac{1}{|\zeta|^N}\frac{1}{\ell(I)^N}\right\}|\zeta|^2\, d\zeta \\
\approx & \frac{1}{\lambda^{\frac{n-1}{2}+2}}\frac{\ell(J)^\kappa}{\ell(I)^2}\sqrt{|I|}\left(\int_0^{\frac{1}{\ell(I)}} r^2 r^{n-2}\, dr + \frac{1}{\ell(I)^N}\int_{\frac{1}{\ell(I)}}^\infty \frac{1}{r^N}r^2 r^{n-2}\, dr\right).
\end{aligned}$$

Choosing $N = n+2$ so the second integral is finite, we get

$$\begin{aligned}
\left|\mathfrak{R}^{(1)}_{\widehat{\varphi^\eta_I},\phi}(\xi)\right| &\lesssim \frac{1}{\lambda^{\frac{n-1}{2}+2}}\frac{\ell(J)^\kappa}{\ell(I)^2}\sqrt{|I|}\frac{1}{\ell(I)^{n+1}} \approx \frac{1}{\left(\operatorname{dist}(0,J)\,\ell(I)^2\right)^{\frac{n-1}{2}+2}}\frac{\ell(I)^{n+3}}{\ell(I)^{n+3}}\ell(J)^\kappa\sqrt{|I|} \\
&\lesssim 2^{-d\left(\frac{n-1}{2}+2\right)}\ell(J)^\kappa\sqrt{|I|},
\end{aligned}$$

if we take $\kappa \geq N = n+2$.

Remark 38. *This error estimate is the same estimate as that for the main term, but with an* additional *small factor of* 2^{-2d}.

Combining the two estimates for the principle term and the remainder term, we have

$$\begin{aligned}
\left|\left\langle Th^{n-1,\eta}_{I;\kappa}, h^{n,\eta}_{J;\kappa}\right\rangle\right| &\leq \left|\left\langle \mathfrak{P}_{\widehat{\varphi^\eta_I},\phi}, h^{n,\eta}_{J;\kappa}\right\rangle\right| + \left|\left\langle \mathfrak{R}^{(1)}_{\widehat{\varphi^\eta_I},\phi}, h^{n,\eta}_{J;\kappa}\right\rangle\right| \\
&\lesssim 2^{-d\frac{n-1}{2}}2^{-|k|\kappa}\sqrt{|I|\,|J|} + 2^{-d\frac{n+3}{2}}2^{-|k|\kappa}\sqrt{|I|\,|J|},
\end{aligned}$$

when $k \leq 0$, $d \geq 0$, and $\kappa \geq n+2$. We record this as

$$(6.17) \qquad \left|\left\langle Th^{n-1,\eta}_{I;\kappa}, h^{n,\eta}_{J;\kappa}\right\rangle\right| \lesssim 2^{-d\frac{n-1}{2}}\ 2^{-|k|\kappa}\ \sqrt{|I|\,|J|}.$$

Next, we will use the estimate (6.17), in the argument we used above to bound $\mathsf{B}^{0,d}_{\text{below}}(f,g)$, to show that there is $\delta > 0$ such that for all $p > \frac{2n}{n-1}$,

$$\left|\mathsf{B}^{k,d}_{\text{below}}(f,g)\right| \lesssim 2^{-|k|\delta}2^{-|d|\delta}\|f\|_{L^p}\|g\|_{L^{p'}}, \qquad \text{for all } k \leq 0, d \geq 0.$$

Of course we now have $d \geq 0$ instead of the opposite inequality $d \leq 0$ used in the previous argument, but we will see that much of the geometry of the decomposition remains the same.

For $k\leq 0$ and $d\geq 0$, the estimates (6.17) imply,

$$\begin{aligned}
\left|\mathsf{B}_{\text{below}}^{k,d}\left(f,g\right)\right| &\equiv \left|\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left\langle T\bigtriangleup_{I;\kappa}^{n-1,\eta}f,\bigtriangleup_{J;\kappa}^{n,\eta}g\right\rangle\right| = \left|\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left\langle Th_{I;\kappa}^{n-1,\eta}f,h_{J;\kappa}^{n,\eta}g\right\rangle_{\omega}\left\langle f,h_{I;\kappa}^{n-1,\eta}\right\rangle\left\langle g,h_{J;\kappa}^{n,\eta}\right\rangle\right| \\
&\lesssim \sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left|\left\langle Th_{I;\kappa}^{n-1,\eta}f,h_{J;\kappa}^{n,\eta}g\right\rangle\right|\left\{\frac{1}{\sqrt{|I|}}\int_{I_\eta}\left|\bigtriangleup_{I;\kappa}^{n-1,\eta}f\left(x\right)\right|dx\right\}\left\{\frac{1}{\sqrt{|J|}}\int_{J_\eta}\left|\bigtriangleup_{J;\kappa}^{n,\eta}g\left(\xi\right)\right|d\xi\right\} \\
&\lesssim \int_{\mathbb{R}^n}\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\frac{\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|}{\sqrt{|I|}\sqrt{|J|}}\left\{\int_{I_\eta}\left|\bigtriangleup_{I;\kappa}^{n-1,\eta}f\left(x\right)\right|dx\right\}\left|\bigtriangleup_{J;\kappa}^{n,\eta}g\left(\xi\right)\right|d\xi \\
&\lesssim 2^{-d\frac{n-1}{2}}2^{-|k|\kappa}\int_{\mathbb{R}^n}\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left\{\int_{I_\eta}\left|\bigtriangleup_{I;\kappa}^{n-1,\eta}f\left(x\right)\right|dx\right\}\left|\bigtriangleup_{J;\kappa}^{n,\eta}g\left(\xi\right)\right|d\xi
\end{aligned}$$

which is at most

$$\begin{aligned}
&2^{-d\frac{n-1}{2}}2^{-|k|\kappa}\int_{\mathbb{R}^n}\sqrt{\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left(\int_{I_\eta}\left|\bigtriangleup_{I;\kappa}^{n-1,\eta}f\left(x\right)\right|dx\right)^2}\sqrt{\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left|\bigtriangleup_{J;\kappa}^{n,\eta}g\left(\xi\right)\right|^2}d\xi \\
\lesssim\ &2^{-d\frac{n-1}{2}}2^{-|k|\kappa}\left(\int_{\mathbb{R}^n}\left(\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left(\int_{I_\eta}\left|\bigtriangleup_{I;\kappa}^{n-1,\eta}f\left(x\right)\right|dx\mathbf{1}_J\left(\xi\right)\right)^2\right)^{\frac{p}{2}}d\xi\right)^{\frac{1}{p}} \\
&\times\left(\int_{\mathbb{R}^n}\left(\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left|\bigtriangleup_{J;\kappa}^{n,\eta}g\left(\xi\right)\right|^2\right)^{\frac{p'}{2}}d\xi\right)^{\frac{1}{p'}} \\
\equiv\ &2^{-d\frac{n-1}{2}}2^{-|k|\kappa}\Gamma_1\Gamma_2.
\end{aligned}$$

We have

$$\Gamma_2^{p'}=\int_{\mathbb{R}^n}\left(\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left|\bigtriangleup_{J;\kappa}^{n,\eta}g\left(x\right)\right|^2\right)^{\frac{p'}{2}}dx=\int_{\mathbb{R}^n}\left(\sum_{J\in\mathcal{D}}\left(\sum_{I\in\mathcal{G}:\ (I,J)\in\mathcal{P}_0^{k,d}}1\right)\left|\bigtriangleup_{J;\kappa}^{n,\eta}g\left(x\right)\right|^2\right)^{\frac{p'}{2}}dx,$$

and now we repeat some of the geometric constructions relating to $\mathcal{P}_0^{k,d}$ from before. Fix $J\in\mathcal{D}$ and let $I_J\in\mathcal{G}$ satisfy

$$c_n\pi_1\left(J\right)\leq\ell\left(I_J\right)\leq\pi_1\left(J\right)\text{ and }I_J\subset\pi_1\left(J\right),$$

where $\pi_1\left(J\right)$ is the spherical projection J onto $\mathbb{S}^{n-1}$, and where $c_n>0$ is chosen small enough that such a cube I_J exists. Now $(I,J)\in\mathcal{P}_0^{k,d}$ if and only if

$$J\subset\mathcal{K}\left(I\right)\text{ and }\frac{2^{d-1}}{\ell\left(I\right)^2}\leq\operatorname{dist}\left(0,J\right)\leq\frac{2^{d+1}}{\ell\left(I\right)^2},$$

which is essentially equivalent to

$$I\supset\pi_1J\supset I_J\text{ and }\sqrt{\frac{2^{d-1}}{2\operatorname{dist}\left(0,J\right)}}\leq\ell\left(I\right)\leq\sqrt{\frac{2^{d+1}}{\operatorname{dist}\left(0,J\right)}}.$$

Thus just as in the previous argument, the set of cubes $I\in\mathcal{G}\left[U\right]$ with $(I,J)\in\mathcal{P}_0^{k,d}$ is contained in the finite tower of dyadic cubes $\left\{\pi^{(k)}I_J\right\}_{k=d-A}^{d+A}$ for some fixed $A\in\mathbb{N}$. It follows that $\sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{C}_0^{0,0}}1\leq 2A$ and so

$$\Gamma_2^{p'} = \int_{\mathbb{R}^n} \left(\sum_{(I,J)\in\mathcal{P}_0^{0,0}} \left| \triangle_{J;\kappa}^{n,\eta} g(x) \right|^2 \right)^{\frac{p'}{2}} dx \leq \int_{\mathbb{R}^n} \left(\sum_{J\in\mathcal{D}} 2A \left| \triangle_{J;\kappa}^{n,\eta} g(x) \right|^2 \right)^{\frac{p'}{2}} dx \lesssim \|g\|_{L^{p'}}^{p'} .$$

We see that on the other hand, since the cubes J in $\mathcal{D}_k$ are pairwise disjoint with measure 2^{kn},

$$\begin{aligned}
\Gamma_1^p &= \int_{\mathbb{R}^n} \left(\sum_{(I,J)\in\mathcal{P}_0^{k,d}} \left(\int_{I_\eta} \left| \triangle_{I;\kappa}^{n-1,\eta} f(x) \right| dx \right)^2 \mathbf{1}_{J_\eta}(\xi) \right)^{\frac{p}{2}} d\xi \\
&\approx \int_{\mathbb{R}^n} \left(\sum_{J\in\mathcal{D}_k} \left\{ \sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_0^{k,d}} \left(\int_{I_\eta} \left| \triangle_{I;\kappa}^{n-1,\eta} f(x) \right| dx \right)^2 \right\} \mathbf{1}_{J_\eta}(\xi) \right)^{\frac{p}{2}} d\xi \\
&= \int_{\mathbb{R}^n} \sum_{J\in\mathcal{D}_k} \left\{ \sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_0^{k,d}} \left(\int_{I_\eta} \left| \triangle_{I;\kappa}^{n-1,\eta} f(x) \right| dx \right)^2 \right\}^{\frac{p}{2}} \mathbf{1}_{J_\eta}(\xi)\, d\xi \\
&\approx \sum_{J\in\mathcal{D}_k} 2^{kn} \left(\sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_0^{k,d}} \left(\int_{I_\eta} \left| \triangle_{I;\kappa}^{n-1,\eta} f(x) \right| dx \right)^2 \right)^{\frac{p}{2}} .
\end{aligned}$$

Now for each fixed $J \in \mathcal{D}_k$ we have with A as above,

$$\begin{aligned}
&\left(\sum_{I\in\mathcal{G}:\ (I,J)\in\mathcal{P}_0^{k,d}} \left(\int_{I_\eta} \left| \triangle_{I;\kappa}^{n-1,\eta} f(x) \right| dx \right)^2 \right)^{\frac{p}{2}} \leq \left(\sum_{s=d-A}^{d+A} \left(\int_{\pi^{(s)}(I_J)_\eta} \left| \triangle_{\pi^{(s)}(I_J);\kappa}^{n-1,\eta} f(x) \right| dx \right)^2 \right)^{\frac{p}{2}} \\
\leq\ & (2A)^{\frac{p}{2}-1} \sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_0^{k,d}} \left(\int_{I_\eta} \left| \triangle_{I;\kappa}^{n-1,\eta} f(x) \right| dx \right)^p \approx \sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_0^{k,d}} \left(\int_{I_\eta} \left| \triangle_{I;\kappa}^{n-1,\eta} f(x) \right| dx \right)^p .
\end{aligned}$$

Altogether then,

$$\begin{aligned}
\Gamma_1^p &\lesssim\gtrsim \sum_{J\in\mathcal{D}_k} 2^{kn} \sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_0^{k,d}} \left(\int_{I_\eta} \left| \triangle_{I;\kappa}^{n-1,\eta} f(x) \right| dx \right)^p \\
&\leq \sum_{J\in\mathcal{D}_k} 2^{kn} \sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_0^{k,d}} |I|^{\frac{p}{2}} \left(\int_{I_\eta} \left| \triangle_{I;\kappa}^{n-1,\eta} f(x) \right|^2 dx \right)^{\frac{p}{2}} \\
&= 2^{kn} \sum_{I\in\mathcal{G}[U]} \left(\sum_{J\in\mathcal{D}_k:\ (I,J)\in\mathcal{P}_0^{k,d}} 1 \right) |I|^p \left(\frac{1}{|I_\eta|} \int_{I_\eta} \left| \triangle_{I;\kappa}^{n-1,\eta} f(x) \right|^2 dx \right)^{\frac{p}{2}} ,
\end{aligned}$$

and since

$$\begin{aligned}
&\# \left\{ J \in \mathcal{D}_k :\ (I,J) \in \mathcal{P}_0^{k,d} \right\} \\
\approx\ & 2^{-kn} |\mathcal{K}_d(I)| \approx 2^{-kn} \left(\frac{2^d}{\ell(I)^2} \ell(I) \right)^{n-1} \frac{2^d}{\ell(I)^2} = 2^{-kn} \frac{2^{dn}}{\ell(I)^{n+1}} = 2^{-kn} 2^{dn} \left(\frac{1}{|I|} \right)^{\frac{n+1}{n-1}} ; \\
&\text{where } \mathcal{K}_d(I) \equiv \left\{ J \subset \mathcal{K}(I) : \frac{2^{d-1}}{\ell(I)^2} \leq \operatorname{dist}(0,J) \leq \frac{2^{d+1}}{\ell(I)^2} \right\},
\end{aligned}$$

we have that

$$\begin{aligned}\Gamma_1^p &\lesssim 2^{kn}\sum_{I\in\mathcal{G}[U]}\left(\#\left\{J\in\mathcal{D}_k:\ (I,J)\in\mathcal{P}_0^{k,d}\right\}\right)|I|^p\left(\frac{1}{|I_\eta|}\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|^2dx\right)^{\frac{p}{2}}\\ &\lesssim 2^{kn}2^{-kn}2^{dn}\sum_{I\in\mathcal{G}[U]}|I|^{p-\frac{n+1}{n-1}}\left(\frac{1}{|I_\eta|}\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|^2dx\right)^{\frac{p}{2}}\\ &= 2^{dn}\int_S\sum_{I\in\mathcal{G}[U]}|I|^{p-\frac{n+1}{n-1}-1}\left(\frac{1}{|I_\eta|}\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|^2dx\right)^{\frac{p}{2}}\mathbf{1}_I(z)\,dz\lesssim 2^{dn}\|f\|_{L^p}^p\ ,\end{aligned}$$

provided $p\geq\frac{2n}{n-1}$, using the the square function estimate (1.17) as in (6.7) above. Thus we have proved,

$$\begin{aligned}\left|\mathsf{B}_{\text{below}}^{k,d}(f,g)\right| &\lesssim 2^{-d\frac{n-1}{2}}2^{-|k|\kappa}\left(2^{dn}\right)^{\frac{1}{p}}\|f\|_{L^p}\|g\|_{L^{p'}}\\ &\lesssim 2^{-d\left(\frac{n-1}{2}-\frac{n}{p}\right)}2^{-|k|\kappa}\|f\|_{L^p}\|g\|_{L^{p'}}\ ,\qquad\text{for } k\leq 0, d\geq 0,\end{aligned}$$

and so

$$\sum_{k\leq 0}\sum_{d\geq 0}\left|\mathsf{B}_{\text{below}}^{k,d}(f,g)\right|\lesssim\sum_{k\leq 0}2^{-|k|\kappa}\sum_{d\geq 0}2^{-d\left(\frac{n-1}{2}-\frac{n}{p}\right)}\|f\|_{L^p}\|g\|_{L^{p'}}\lesssim\|f\|_{L^p}\|g\|_{L^{p'}}\ ,$$

provided $p>\frac{2n}{n-1}$, and $\kappa\geq 1$. Note that we only needed *strict* inequality $p>\frac{2n}{n-1}$ in this last line. Moreover, the previous lines of argument can be simplified when $p>\frac{2n}{n-1}$ - see Subsubsection 9.3.1.

6.4. **Subforms with $k\geq 0, d\geq 0$.** We take both k and d to be nonnegative, and begin with the radial integration by parts formula (4.7) to obtain,

$$\begin{aligned}\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle &= \int_{(0,\infty)}\int_{\mathbb{R}^{n-1}}\left\{\int_{\mathbb{R}^{n-1}}e^{i\lambda\phi(x,y)}\frac{\varphi_I^\eta(x)}{\phi(x,y)^Z}dx\right\}\partial_\lambda^Z\widetilde{\psi}_J^\eta(y,\lambda)\,dyd\lambda\\ &= \int_{(0,\infty)}\int_{\mathbb{R}^{n-1}}\mathcal{I}_{\widetilde{\varphi_I^\eta},\phi}(y,\lambda)\,\partial_\lambda^Z\widetilde{\psi}_J^\eta(y,\lambda)\,dyd\lambda,\end{aligned}$$

where

$$\mathcal{I}_{\widetilde{\varphi_I^\eta},\phi}(y,\lambda)=\int_{\mathbb{R}^{n-1}}e^{i\lambda\phi(x,y)}\frac{\varphi_I^\eta(x)}{\phi(x,y)^Z}dx$$

which is an oscillatory term having the form of (4.17), but with amplitude

$$\widetilde{\varphi_I^\eta}(x,y)=\frac{\varphi_I^\eta(x)}{\phi(x,y)^Z},$$

in place of $\varphi_I^\eta(x)$, which is then paired with the function

$$\partial_\lambda^Z\widetilde{\psi}_J^\eta(y,\lambda)=\partial_\lambda^Z h_{J;\kappa}^{n,\eta}\left(\lambda y,\lambda\sqrt{1-|y|^2}\right)\frac{\lambda^{n-1}}{\sqrt{1-|y|^2}}$$

in place of $\widetilde{\psi}_J^\eta(y,\lambda)$, and where we can take $Z\in\mathbb{N}$ to be a large positive integer depending only on n.

Now we proceed by treating the integral

$$\int_{(0,\infty)}\int_{\mathbb{R}^{n-1}}\mathcal{I}_{\widetilde{\varphi_I^\eta},\phi}(y,\lambda)\,\partial_\lambda^Z\widetilde{\psi}_J^\eta(y,\lambda)\,dyd\lambda$$

as in the previous case where $k\leq 0$ and $d\geq 0$, but with the new amplitudes $\widetilde{\varphi_I^\eta}$ and pairing functions $\partial_\lambda^Z\widetilde{\psi}_J^\eta(y,\lambda)$ as above. The end result that we will obtain below is the estimate,

$$\left|\mathsf{B}_{\text{below}}^{k,d}(f,g)\right|\lesssim 2^{-d\delta}2^{-k\delta}\|f\|_{L^p}\|g\|_{L^{p'}}\ ,\qquad\text{for } k\geq 0, d\geq 0,\tag{6.18}$$

for some $\delta>0$.

Indeed, we apply Theorem 28 to $\mathcal{I}_{\widetilde{\varphi_I^\eta},\phi}(y,\lambda) = \mathfrak{P}_{\widetilde{\varphi_I^\eta},\phi}(y,\lambda) + \mathfrak{R}^{(1)}_{\widetilde{\varphi_I^\eta},\phi}(y,\lambda)$ and first note that

$$\mathfrak{P}_{\widetilde{\varphi_I^\eta},\phi}(y,\lambda) = \left(\frac{2\pi}{\lambda}\right)^{\frac{n-1}{2}} \frac{e^{i\,\mathrm{sgn}\left[\partial_x^2\phi(X(y),y)\right]\frac{\pi}{4}+\lambda\phi(X(y),y)}}{\sqrt{|\det B(y)|}}\widetilde{\varphi_I^\eta}(X(y)),$$

and arguing as above, we get

$$\left|\int_{(0,\infty)} \mathfrak{P}_{\widetilde{\varphi_I^\eta},\phi}(y,\lambda)\,\partial_\lambda^Z\widetilde{\psi}_J^\eta(y,\lambda)\,dyd\lambda\right| \lesssim 2^{-d\frac{n-1}{2}}2^{-kZ}\sqrt{|I|\,|J|}.$$

As for the remainder term $\mathfrak{R}^{(1)}_{\widetilde{\varphi_I^\eta},\phi}(y,y_J,\lambda)$, we again invoke the argument of C. Rios to obtain from (6.16) with $\kappa=0$ that

$$\text{(6.19a)}\qquad \left|\left\langle \mathfrak{R}^{(1)}_{\widetilde{\varphi_I^\eta},\phi}(y,\lambda), \partial_\lambda h^{n,\eta}_{J;\kappa}\right\rangle\right| \leq \int\left|\mathfrak{R}^{(1)}_{\widetilde{\varphi_I^\eta},\phi}(\xi)\,\partial_\lambda h^{n,\eta}_{J;\kappa}(\xi)\right|d\xi \leq \left\|\mathfrak{R}^{(1)}_{\widetilde{\varphi_I^\eta},\phi}\right\|_{L^\infty}\left\|\partial_\lambda h^{n,\eta}_{J;\kappa}\right\|_{L^\infty}|J|$$
$$\lesssim\ 2^{-d\left(\frac{n-1}{2}+2\right)}2^{-kZ}\sqrt{|I|}\sqrt{|J|} \leq 2^{-d\frac{n-1}{2}}2^{-kZ}\sqrt{|I|\,|J|},$$

where we have discarded the small factor 2^{-2d}.

6.4.1. *The square function estimates.* From above, we have the estimate,

$$\left|\int_{(0,\infty)}\int_S \mathcal{I}_{\widetilde{\varphi_I^\eta},\phi}(y_J,\lambda)\,\partial_\lambda^Z\widehat{\psi}_J^\eta(y,\lambda)\,dyd\lambda\right| \lesssim 2^{-d\frac{n-1}{2}}2^{-kZ}\sqrt{|I|\,|J|}.$$

Now we apply the square function arguments to obtain (6.18) for some $\delta>0$ by choosing Z sufficiently large depending on n. Indeed, following the argument in the above subsection, we have

$$\left|\mathsf{B}^{k,d}_{\mathrm{below}}(f,g)\right| \lesssim 2^{-d\frac{n-1}{2}}2^{-kZ}\int_{\mathbb{R}^n}\sqrt{\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left(\int_{I_\eta}\left|\triangle^{n-1,\eta}_{I;\kappa}f(x)\right|dx\right)^2}\sqrt{\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left|\triangle^{n,\eta}_{J;\kappa}g(\xi)\right|^2}d\xi$$
$$\lesssim\ 2^{-d\frac{n-1}{2}}2^{-kZ}\left(\int_{\mathbb{R}^n}\left(\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left(\int_{I_\eta}\left|\triangle^{n-1,\eta}_{I;\kappa}f(x)\right|dx\mathbf{1}_{J_\eta}(\xi)\right)^2\right)^{\frac{p}{2}}d\xi\right)^{\frac{1}{p}}$$
$$\times\left(\int_{\mathbb{R}^n}\left(\sum_{(I,J)\in\mathcal{P}_0^{k,d}}\left|\triangle^{n,\eta}_{J;\kappa}g(\xi)\right|^2\right)^{\frac{p'}{2}}d\xi\right)^{\frac{1}{p'}}$$
$$\equiv\ 2^{-d\frac{n-1}{2}}2^{-kZ}\Gamma_1\Gamma_2.$$

and $\sum_{I\in\mathcal{G}:\ (I,J)\in\mathcal{C}_0^{0,0}}1\leq 2A$, which together give,

$$\Gamma_2^{p'} = \int_{\mathbb{R}^n}\left(\sum_{(I,J)\in\mathcal{P}_0^{0,0}}\left|\triangle^{n,\eta}_{J;\kappa}g(x)\right|^2\right)^{\frac{p'}{2}}dx \leq \int_{\mathbb{R}^n}\left(\sum_{J\in\mathcal{D}}2A\left|\triangle^{n,\eta}_{J;\kappa}g(x)\right|^2\right)^{\frac{p'}{2}}dx \lesssim \|g\|^{p'}_{L^{p'}},$$

by the square function estimate (1.17).

We also have

$$\Gamma_1^p = 2^{kn}\sum_{J\in\mathcal{D}_k}\left(\sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_0^{k,d}}\left(\int_{I_\eta}\left|\triangle^{n-1,\eta}_{I;\kappa}f(x)\right|dx\right)^2\right)^{\frac{p}{2}},$$

and since $k \geq 0$, we obtain that $\#\left\{J \in \mathcal{D}_k : (I,J) \in \mathcal{P}_0^{k,d}\right\} \lesssim 2^{-kn}$, which yields

$$\begin{aligned}
\Gamma_1^p &\lesssim 2^{kn} \sum_{I \in \mathcal{G}[U]} \left(\#\left\{J \in \mathcal{D}_k : (I,J) \in \mathcal{P}_0^{k,d}\right\}\right) |I|^p \left(\frac{1}{|I_\eta|} \int_{I_\eta} \left|\triangle_{I;\kappa}^{n-1,\eta} f(x)\right|^2 dx\right)^{\frac{p}{2}} \\
&\lesssim 2^{kn} 2^{-kn} 2^{dn} \sum_{I \in \mathcal{G}[U]} |I|^{p-\frac{n+1}{n-1}} \left(\frac{1}{|I_\eta|} \int_{I_\eta} \left|\triangle_{I;\kappa}^{n-1,\eta} f(x)\right|^2 dx\right)^{\frac{p}{2}} \\
&= 2^{dn} \int_S \sum_{I \in \mathcal{G}[U]} |I|^{p-\frac{n+1}{n-1}-1} \left(\frac{1}{|I_\eta|} \int_{I_\eta} \left|\triangle_{I;\kappa}^{n-1,\eta} f(x)\right|^2 dx\right)^{\frac{p}{2}} \mathbf{1}_I(z)\, dz \lesssim 2^{dn} \|f\|_{L^p}^p ,
\end{aligned}$$

just as before, by the square function estimate (1.17), provided $p \geq \frac{2n}{n-1}$.

Altogether then we have

$$\left|\mathsf{B}_{\text{below}}^{k,d}(f,g)\right| \lesssim 2^{-d\frac{n-1}{2}} 2^{-kZ} \Gamma_1 \Gamma_2 \lesssim 2^{-d\left(\frac{n-1}{2}-\frac{n}{p}\right)} 2^{-kZ} \|f\|_{L^p} \|g\|_{L^{p'}} ,$$

which implies (6.18) with

$$\delta \equiv \min\left\{\frac{n-1}{2} - \frac{n}{p}, Z\right\} > 0,$$

provided $p > \frac{2n}{n-1}$ and $Z \geq 1$. Finally, summing in $k, d \geq 0$, we obtain

$$\sum_{k \geq 0} \sum_{d \geq 0} \left|\mathsf{B}_{\text{below}}^{k,d}(f,g)\right| \leq \sum_{k \geq 0} \sum_{d \geq 0} 2^{-d\delta} 2^{-k\delta} \|f\|_{L^p} \|g\|_{L^{p'}} \lesssim \|f\|_{L^p} \|g\|_{L^{p'}} .$$

6.5. **Wrapup.** Combining the estimates from all four subsections above yields the desired bound,

$$\left|\mathsf{B}_{\text{below}}(f,g)\right| \lesssim \|f\|_{L^p} \|g\|_{L^{p'}} , \qquad p > \frac{2n}{n-1},$$

in fact the stronger bound (6.1).

Remark 39. *Apart from the standard reduction* **??** *in Section 3, the* strict *inequality* $p > \frac{2n}{n-1}$ *was used only in bounding the below form for large* d*. We will also use* $p > \frac{2n}{n-1}$ *for probabilistic control of the disjoint form, but only* $p > 1$ *for controlling the above form* $\mathsf{B}_{\text{above}}(f,g)$*, to which we turn next.*

7. Control of the above form

Next we control the above form,

$$\mathsf{B}_{\text{above}}(f,g) \equiv \sum_{(I,J) \in \mathcal{R}} \left\langle T h_{I;\kappa}^{n-1,\eta}, h_{J;\kappa}^{n,\eta} \right\rangle ,$$

where

$$\mathcal{R} \equiv \left\{(I,J) \in \mathcal{G}[U] \times \mathcal{D} : \Phi(I) \subset \pi_{\tan}(C_{\text{pseudo}} J)\right\} .$$

For this form, we will use the pigeonholed parameter $k = \log_2 \ell(J)$ already used in the below subforms, together with a new parameter $r = \log_2 \frac{\ell(\pi_{\tan} J)}{\ell(I)}$, measuring the ratio of the side lengths of I and $\pi_{\tan} J$. Note that for fixed k and r, and a fixed cube $I \in \mathcal{G}$, there is at most a bounded number of cubes $J \in \mathcal{D}$ satisfying the pigeonholed properties $\ell(J) = 2^k$ and $\frac{\ell(\pi_{\tan} J)}{\ell(I)} = 2^r$ such that $(I,J) \in \mathcal{R}$. This fact dictates that we arrange our square function decompositions relative to the cubes I in the grid $\mathcal{G}$ (rather than to cubes J in $\mathcal{D}$ as as in $\mathsf{B}_{\text{below}}(f,g)$) in the arguments below.

To achieve geometric decay in both of these parameters, we will use the high order moment vanishing principle of decay for the Alpert wavelets $h_{I;\kappa}^{n-1,\eta}$ in S for decay in r, an integration by parts in the radial Fourier variable for decay in $k \geq 0$, and the high order moment vanishing principle of decay for the Alpert wavelets $h_{J;\kappa}^{n,\eta}$ for decay in $k \leq 0$. The stationary phase estimate in Theorem 28 is not needed for the form $\mathsf{B}_{\text{above}}(f,g)$.

In fact we will prove the stronger result that the sublinear form

$$|\mathsf{B}_{\text{above}}|(f,g) \equiv \sum_{(I,J)\in\mathcal{R}} \left|\left\langle T_S h^{n-1,\eta}_{I;\kappa}, h^{n,\eta}_{J;\kappa}\right\rangle\right|$$

satisfies

$$|\mathsf{B}_{\text{above}}|(f,g) \lesssim \|f\|_{L^p}\|g\|_{L^{p'}}, \qquad \text{for } p > \frac{2n}{n-1}. \tag{7.1}$$

Here is the decomposition of $\mathcal{R}$ we will use:

$$\begin{aligned}\mathcal{R} &= \bigcup_{k\in\mathbb{Z}}\bigcup_{r=1}^{\infty}\mathcal{R}^{k,r}, \text{ where for all } k\in\mathbb{Z} \text{ and } r\in\mathbb{N}, \\ \mathcal{R}^{k,r} &\equiv \left\{(I,J)\in\mathcal{R} : \ell(J)=2^k, \text{ and } \ell(\pi_{\tan}J)\approx 2^r\ell(I)\right\}.\end{aligned} \tag{7.2}$$

First we reduce matters to consideration of cubes J that are disjoint from a large cube $\left[-2^M,2^M\right]^n$ centered at the origin, which will permit the manipulations used below.

7.1. **Reduction to far away dyadic cubes.** We now dispense with the first set of trivial pairs $(I,J)\in\mathcal{R}$, namely those for which $J\subset\left[-2^M,2^M\right]^n$ for some fixed large positive integer M. This can be achieved by splitting the function g into

$$g = \mathbf{1}_{[-2^M,2^M]^n}g + \mathbf{1}_{\mathbb{R}^n\setminus[-2^M,2^M]^n}g = g_1+g_2,$$

and noting that

$$|\langle Tf,g_1\rangle| \lesssim \|f\|_{L^1}\|g_1\|_{L^1} \lesssim \|f\|_{L^p}2^{Mnp}\|g_1\|_{L^{p'}}, \qquad 1<p<\infty.$$

Then we may assume that g is supported outside $\left[-2^M,2^M\right]$, and it follows that $\triangle^{n,\eta}_{J;\kappa}f = \left\langle f, h^n_{J;\kappa}\right\rangle h^{n,\eta}_{J;\kappa}$ vanishes for $J\subset\left[-2^M,2^M\right]^n$.

Next we deal with the slightly less trivial case of dyadic cubes J that have the origin as one of their vertices. These cubes are contained in 2^n towers of dyadic cubes, and we will derive here the bound corresponding to the tower $\{J_k\}_{k=M}^{\infty}$ where $J_k=\left[0,2^k\right]^n$, the other cases being similar. First we note that

$$\left(\frac{1}{-ix_n}\mathbf{e}_n\cdot\partial_\xi\right)^N e^{-ix\cdot\xi} = e^{-ix\cdot\xi} \text{ for all } N,$$

and so integrating by parts N times gives,

$$\begin{aligned}\left\langle Tf,\triangle^{n,\eta}_{J_k}g\right\rangle &= \int_{\mathbb{R}^n}\int_{\Phi(S)} f(z)\,e^{-iz\cdot\xi}d\sigma_{n-1}(z)\,\triangle^{n,\eta}_{J_k}g(\xi)\,d\xi \\ &= \int_{\Phi(S)}\left\{\int_{\mathbb{R}^n} e^{-iz\cdot\xi}h^{n,\eta}_{J_k}(\xi)\left\langle g,h^{n,\eta}_{J_k}\right\rangle d\xi\right\}d\sigma_{n-1}(z) \\ &= i^N\left\langle g,h^{n,\eta}_{J_k}\right\rangle\int_S\left\{\int_{\mathbb{R}^n} e^{-iz\cdot\xi}\left(\mathbf{e}_n\cdot\partial_\xi\right)^N h^{n,\eta}_{J_k}g(\xi)\,d\xi\right\}\left(\frac{1}{z_n}\right)^N f(z)\,d\sigma_{n-1}(z),\end{aligned}$$

and then

$$\begin{aligned}&\sum_{k=N}^{\infty}\left|\left\langle Tf,\triangle^{n,\eta}_{J_k}g\right\rangle\right| \lesssim \sum_{k=N}^{\infty}\left|\left\langle g,h^{n,\eta}_{J_k}\right\rangle\right|\int_S\left(\frac{1}{\eta\ell(J_k)}\right)^N\sqrt{|J_k|}\left(\frac{1}{x_n}\right)^N f(x)\,dx \\ \le\ &\left(\frac{1}{\eta}\right)^N\sum_{k=N}^{\infty}\left|\left\langle g,h^{n,\eta}_{J_k}\right\rangle\right|\ell(J_k)^{\frac{n}{2}-N}\|f\|_{L^1} = \left(\frac{1}{\eta}\right)^N\|f\|_{L^1}\int_{\mathbb{R}^n}\sum_{k=N}^{\infty}\left(\left|\left\langle g,h^{n,\eta}_{J_k}\right\rangle\right|\frac{1}{\sqrt{|J_k|}}\right)\ell(J_k)^{-N}\mathbf{1}_{J_k}(z)\,dz \\ \le\ &\left(\frac{1}{\eta}\right)^N\|f\|_{L^1}\int_{\mathbb{R}^n}\left(\sum_{k=N}^{\infty}\left(\left|\left\langle g,h^{n,\eta}_{J_k}\right\rangle\right|\frac{1}{\sqrt{|J_k|}}\right)^2\mathbf{1}_{J_k}(z)\right)^{\frac12}\left(\sum_{k=N}^{\infty}\ell(J_k)^{-2N}\mathbf{1}_{J_k}(z)\right)^{\frac12}dz \\ \le\ &\left(\frac{1}{\eta}\right)^N\|f\|_{L^1}\left(\int_{\mathbb{R}^n}\left(\sum_{k=N}^{\infty}\frac{\left|\left\langle g,h^{n,\eta}_{J_k}\right\rangle\right|^2}{|J_k|}\mathbf{1}_{J_k}(z)\right)^{\frac{p'}{2}}dz\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^n}\left(\sum_{k=N}^{\infty}\ell(J_k)^{-2N}\mathbf{1}_{J_k}(z)\right)^{\frac{p}{2}}dz\right)^{\frac1p}.\end{aligned}$$

Thus we obtain

$$\sum_{k=N}^{\infty}\left|\left\langle T_S f,\triangle_{J_k;\kappa}^{n,\eta}g\right\rangle\right|\leq C_{p,N}\left(\frac{1}{\eta}\right)^N\left\|f\right\|_{L^1}\left\|g\right\|_{L^{p'}}\leq C_{p,N}\left(\frac{1}{\eta}\right)^N\left\|f\right\|_{L^p}\left\|g\right\|_{L^{p'}}\ ,\qquad 1<p<\infty,$$

using the equivalence (2.1) of square function norms on g, together with the finiteness of the final factor if N is chosen sufficiently large. Indeed, $\left\|g\right\|_{L^{p'}}\approx\left\|\mathcal{S}g\right\|_{L^{p'}}$ where

$$\begin{aligned}\left\|\mathcal{S}g\right\|_{L^{p'}}^{p'} &= \int_{\mathbb{R}^n}\left(\sum_{J\in\mathcal{D}}^{\infty}\left|\triangle_{J;\kappa}^{n,\eta}g\left(z\right)\right|^2\right)^{\frac{p'}{2}}dz=\int_{\mathbb{R}^n}\left(\sum_{J\in\mathcal{D}}^{\infty}\left(\left\langle g,h_{J;\kappa}^{n,\eta}\right\rangle h_{J;\kappa}^{n,\eta}\left(z\right)\right)^2\right)^{\frac{p'}{2}}dz\\ &= \int_{\mathbb{R}^n}\left(\sum_{J\in\mathcal{D}}^{\infty}\left(\left\langle g,h_{J;\kappa}^{n,\eta}\right\rangle\frac{1}{\sqrt{\left|J\right|}}\right)^2\mathbf{1}_{J_\eta}\left(z\right)\right)^{\frac{p'}{2}}dz=\int_{\mathbb{R}^n}\left(\sum_{J\in\mathcal{D}}^{\infty}\frac{\left\langle g,h_{J;\kappa}^{n,\eta}\right\rangle^2}{\left|J\right|}\mathbf{1}_{J_\eta}\left(z\right)\right)^{\frac{p'}{2}},\end{aligned}$$

and for $N>\frac{n}{p}$ we have,

$$\int_{\mathbb{R}^n}\left(\sum_{k=N}^{\infty}\ell\left(J_k\right)^{-2N}\mathbf{1}_{\left(J_k\right)_\eta}\left(z\right)\right)^{\frac{p}{2}}dz=\int_{\mathbb{R}^n}\left(\sum_{k=N}^{\infty}2^{-2Nk}\mathbf{1}_{\left(\left[0,2^k\right]^n\right)_\eta}\left(z\right)\right)^{\frac{p}{2}}dz\lesssim\int_{\mathbb{R}^n}\left(1+\left|z\right|^{-2N}\right)^{\frac{p}{2}}dz<\infty.$$

Definition 40. *Set*

$$\begin{aligned}\mathcal{R}_* &\equiv \left\{\left(I,J\right)\in\mathcal{R}:J\cap\left[-2^N,2^N\right]^n=\emptyset\right\}\\ &= \left\{\left(I,J\right)\in\mathcal{G}\left[U\right]\times\mathcal{D}:\Phi\left(I\right)\subset\pi_{\tan}\left(C_{\text{pseudo}}J\right)\ \textit{and}\ J\cap\left[-2^N,2^N\right]^n=\emptyset\right\},\end{aligned}$$

and with $\mathcal{R}^{k,r}$ as in (7.2),

$$\begin{aligned}\mathcal{R}_*^{k,r} &\equiv \left\{\left(I,J\right)\in\mathcal{R}^{k,r}:J\cap\left[-2^N,2^N\right]^n=\emptyset\right\},\\ \mathcal{R}_*^{r} &\equiv \bigcup_k\mathcal{R}_*^{k,r}\ .\end{aligned}\tag{7.3}$$

Assumption: It is understood from now on that all of the cubes $J\in\mathcal{R}$ considered below in this section satisfy $J\cap\left[-2^N,2^N\right]^n=\emptyset$, i.e. $J\in\mathcal{R}_*$.

7.2. **Pigeonholed subforms.** Using the moment vanishing of the smooth wavelets $h_{I;\kappa}^{n-1,\eta}$, we first show the preliminary estimate that for all $r\in\mathbb{N}$,

$$\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim\ell\left(I\right)^\kappa\ell\left(J\right)^\kappa\sqrt{\left|I\right|\left|J\right|},\qquad\text{for all }\left(I,J\right)\in\mathcal{R}_*^r\text{ when }r\geq1.\tag{7.4}$$

So consider the case $\left(I,J\right)\in\mathcal{R}_*^r$, $r\geq1$. Using (4.10) and (4.11), with c_I denoting the center of I, we have

$$\begin{aligned}&\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle=\int_{\mathbb{R}^n}\int_{\mathbb{R}^{n-1}}e^{-i\Phi(x)\cdot\xi}h_{I;\kappa}^{n-1,\eta}\left(x\right)dxh_{J;\kappa}^{n,\eta}\left(\xi\right)d\xi\\ &= \int_{\mathbb{R}^n}e^{-i\Phi(c_I)\cdot\xi}h_{J;\kappa}^{n,\eta}\left(\xi\right)\left\{\int_{\mathbb{R}^{n-1}}e^{-i[\Phi(x)-\Phi(c_I)]\cdot\xi}h_{I;\kappa}^{n-1,\eta}\left(x\right)dx\right\}d\xi\\ &= \int_{\mathbb{R}^n}e^{-i\Phi(c_I)\cdot\xi}h_{J;\kappa}^{n,\eta}\left(\xi\right)\left\{\int_{\mathbb{R}^{n-1}}\left[\sum_{\ell=0}^{\kappa-1}\frac{\left(-i\xi\cdot\left[\Phi\left(x\right)-\Phi\left(c_I\right)\right]\right)^\ell}{\ell!}+R_\kappa\left(-i\xi\cdot\left[\Phi\left(x\right)-\Phi\left(c_I\right)\right]\right)\right]h_{I;\kappa}^{n-1,\eta}\left(x\right)dx\right\}d\xi.\end{aligned}$$

In order to apply the moment vanishing properties of $h_{I;\kappa}^{n-1,\eta}$, we need to express $\Phi\left(x\right)$ by Taylor's formula as well,

$$\Phi\left(x\right)=\sum_{\ell=0}^{\kappa-1}\frac{\left(\left(x-c_I\right)\cdot\partial_x\right)^\ell}{\ell!}\Phi\left(c_I\right)+\Gamma_\kappa\left(x-c_I\right),$$

and then plug this expression into the previous Taylor formula. The result is that all the terms with a polynomial in x of order less than κ vanish, and we are left with

$$\text{(7.5)}\quad \left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle = \int_{\mathbb{R}^n} e^{-i\Phi(c_I)\cdot\xi}h_{J;\kappa}^{n,\eta}(\xi)\left\{\int_{\mathbb{R}^{n-1}}\Gamma(\xi,x)\,h_{I;\kappa}^{n-1,\eta}(x)\,dx\right\}d\xi$$
$$= \int_{\mathbb{R}^n} e^{-i\Phi(c_I)\cdot\xi}h_{J;\kappa}^{n,\eta}(\xi)\left\{\int_{\mathbb{R}^{n-1}}\left[R_\kappa(-i\xi\cdot[\Phi(x)-\Phi(c_I)])\right]h_{I;\kappa}^{n-1,\eta}(x)\,dx\right\}d\xi$$
$$+\int_{\mathbb{R}^n} e^{-i\Phi(c_I)\cdot\xi}h_{J;\kappa}^{n,\eta}(\xi)\left\{\int_{\mathbb{R}^{n-1}}\left[\Gamma_\kappa(x-c_I)\right]h_{I;\kappa}^{n-1,\eta}(x)\,dx\right\}d\xi$$

where

$$\text{(7.6)}\qquad \Gamma(\xi,x)=R_\kappa(-i\xi\cdot[\Phi(x)-\Phi(c_I)])+\Gamma_\kappa(x-c_I)$$

consists of the remainder term R_κ and a collection of error expressions in $\Gamma_\kappa(\xi,x)$. Because $|x-c_I|\leq|\Phi(x)-\Phi(c_I)|$, these error expressions satisfy the same pointwise bounds as the original remainder term $R_\kappa(-i\xi\cdot[\Phi(x)-\Phi(c_I)])$. Recalling from (4.11) that the remainder term R_κ satisfies $|R_\kappa(ib)|\leq\frac{|b|^\kappa}{(\kappa+1)!}$, and taking absolute values inside the integral, we obtain,

$$\text{(7.7)}\qquad \left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim(\operatorname{dist}(0,J)\,\ell(I)\sin\theta)^\kappa\sqrt{|I|\,|J|},$$

where θ is the angle between ξ and $\Phi(x)-\Phi(c_I)$. In the case at hand where $(I,J)\in\mathcal{R}_*^r$, we have $\theta\approx\ell(\pi_{\tan}J)\approx\frac{\ell(J)}{\operatorname{dist}(0,J)}$, and so

$$\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim\left(\operatorname{dist}(0,J)\,\ell(I)\frac{\ell(J)}{\operatorname{dist}(0,J)}\right)^\kappa\sqrt{|I|\,|J|}\approx\ell(I)^\kappa\,\ell(J)^\kappa\sqrt{|I|\,|J|},\qquad\text{for }(I,J)\in\mathcal{R}_*^r\ ,$$

which proves the preliminary estimate (7.4).

The case $k\leq0$ will be handled by this last estimate alone, since it yields

$$\text{(7.8)}\qquad \left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim\ell(\pi_{\tan}J)^\kappa\left(\frac{\ell(I)}{\ell(\pi_{\tan}J)}\right)^\kappa\ell(J)^\kappa\sqrt{|I|\,|J|}\leq2^{-r\kappa}2^{-|k|\kappa},\qquad\text{for }k\leq0,$$

upon discarding the small factor $\ell(\pi_{\tan}J)^\kappa$.

To handle the case $k\geq0$, we introduce the radial integration by parts principle of decay, that will deliver geometric gain in k. First we observe that $(I,J)\in\mathcal{R}_*$ implies $I\subset\pi_{\tan}(C_{\text{pseudo}}J)$, and so for $\mathbf{v}=\pi_{\tan}c_J$ and for $x\in\pi_{\tan}(C_{\text{pseudo}}J)$ we have

$$\mathbf{v}\cdot\Phi(x)\geq c>0,$$

and

$$\left(\frac{1}{-i\mathbf{v}\cdot\Phi(x)}\mathbf{v}\cdot\partial_\xi\right)^N e^{-i\Phi(x)\cdot\xi}=e^{-ix\cdot\xi}\text{ for all }N.$$

Integrating by parts N times then gives,

$$\text{(7.9)}\qquad \left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle=\int_{\mathbb{R}^n}\int_{\mathbb{R}^{n-1}}h_{I;\kappa}^{n-1,\eta}e^{-i\Phi(x)\cdot\xi}dxh_J^{n,\eta}g(\xi)\,d\xi$$
$$=\int_{\mathbb{R}^{n-1}}\left\{\int_{\mathbb{R}^n}e^{-ix\cdot\xi}h_J^{n,\eta}g(\xi)\,d\xi\right\}h_{I;\kappa}^{n-1,\eta}dx$$
$$=i^N\int_{\mathbb{R}^{n-1}}\left\{\int_{\mathbb{R}^n}e^{-ix\cdot\xi}(\mathbf{v}\cdot\partial_\xi)^N h_J^{n,\eta}(\xi)\,d\xi\right\}\left(\frac{1}{\mathbf{v}\cdot\Phi(x)}\right)^N h_{I;\kappa}^{n-1,\eta}(x)\,dx,$$

and then we have the second preliminary estimate,

$$\text{(7.10)}\qquad \left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim\int_{\mathbb{R}^{n-1}}\left(\frac{1}{\eta\ell(J)}\right)^N\sqrt{|J|}\left(\frac{1}{c}\right)^N\left|h_{I;\kappa}^{n-1,\eta}(x)\right|dx\approx\ell(J)^{-N}\sqrt{|I|\,|J|}.$$

We must now combine these two preliminary estimates in the case $k\geq0$. As usual, to achieve this we iterate the two associated formulas (7.5) and (7.9) *before* taking absolute values inside the resulting integral.

Thus we write,

$$\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle = i^{N}\int_{\mathbb{R}^{n-1}}\left\{\int_{\mathbb{R}^{n}}e^{-i\Phi(x)\cdot\xi}\left(\mathbf{v}\cdot\partial_{\xi}\right)^{N}h_{J;\kappa}^{n,\eta}\left(\xi\right)d\xi\right\}\left(\frac{1}{\mathbf{v}\cdot\Phi\left(x\right)}\right)^{N}h_{I;\kappa}^{n-1,\eta}\left(x\right)dx$$

$$= \int_{\mathbb{R}^{n}}e^{-i\Phi(c_I)\cdot\xi}\left\{\int_{\mathbb{R}^{n-1}}e^{-i[\Phi(x)-\Phi(c_I)]\cdot\xi}h_{I;\kappa}^{n-1,\eta}\left(x\right)\left(\frac{1}{\mathbf{v}\cdot\Phi\left(x\right)}\right)^{N}dx\right\}\left(\mathbf{v}\cdot\partial_{\xi}\right)^{N}h_{J;\kappa}^{n,\eta}\left(\xi\right)d\xi$$

$$= \int_{\mathbb{R}^{n}}e^{-i\Phi(c_I)\cdot\xi}\left\{\int_{\mathbb{R}^{n-1}}\Gamma\left(\xi,x\right)h_{I;\kappa}^{n-1,\eta}\left(x\right)\left(\frac{1}{\mathbf{v}\cdot\Phi\left(x\right)}\right)^{N}dx\right\}\left(\mathbf{v}\cdot\partial_{\xi}\right)^{N}h_{J;\kappa}^{n,\eta}\left(\xi\right)d\xi,$$

where

$$\Gamma\left(\xi,x\right)=R_{\kappa}\left(-i\xi\cdot\left[\Phi\left(x\right)-\Phi\left(c_I\right)\right]\right)+\Gamma_{\kappa}\left(x-c_I\right),$$

is as in (7.6) above, and $\Gamma\left(\xi,x\right)$ satisfies the estimates given there. Now we take absolute values inside the integral, and using the estimates developed above, we obtain the following inequality for $k\geq 0$,

(7.11)
$$\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim\ell\left(I\right)^{\kappa}\ell\left(J\right)^{\kappa}\ell\left(J\right)^{-N}\sqrt{\left|I\right|\left|J\right|}\lesssim\left(\frac{\ell\left(I\right)}{\ell\left(\pi_{\tan}J\right)}\right)^{\kappa}\ell\left(J\right)^{2\kappa-N}\lesssim 2^{-r\kappa}2^{-k(N-2\kappa)}\sqrt{\left|I\right|\left|J\right|}.$$

Combining (7.8) and (7.11) gives

$$\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim 2^{-r\kappa}2^{-\left|k\right|\min\{\kappa,N-2\kappa\}}\sqrt{\left|I\right|\left|J\right|}, \tag{7.12}$$

and with this estimate in hand, we will now prove that for all $N>2\kappa$ and $r\in\mathbb{N}$,

$$\left|\sum_{(I,J)\in\mathcal{R}_{*}^{k,r}}\left\langle T\bigtriangleup_{I;\kappa}^{n-1,\eta}f,\bigtriangleup_{J;\kappa}^{n,\eta}g\right\rangle\right|\lesssim 2^{-r\left(\kappa-\frac{n-1}{2}\right)}2^{-\left|k\right|\min\{\kappa,N-2\kappa\}}\left\|f\right\|_{L^{p}}\left\|g\right\|_{L^{p'}}, \tag{7.13}$$

where $\mathcal{R}_{*}^{k,r}$ is defined in (7.3). Indeed, we have from (7.12) that

$$\sum_{(I,J)\in\mathcal{R}_{*}^{k,r}}\left|\left\langle T\bigtriangleup_{I;\kappa}^{n-1,\eta}f,\bigtriangleup_{J;\kappa}^{n,\eta}g\right\rangle\right|\leq\sum_{(I,J)\in\mathcal{R}_{*}^{k,r}}2^{-r\kappa}2^{-\left|k\right|\min\{\kappa,N-2\kappa\}}\left(\int_{I_{\eta}}\left|\bigtriangleup_{I;\kappa}^{n-1,\eta}f\right|\right)\left(\int_{J_{\eta}}\left|\bigtriangleup_{J;\kappa}^{n,\eta}g\right|\right)$$

$$= 2^{-r\kappa}2^{-\left|k\right|\min\{\kappa,N-2\kappa\}}\int_{\mathbb{R}^{n}}\sum_{(I,J)\in\mathcal{R}_{*}^{k,r}}\left(\int_{J_{\eta}}\left|\bigtriangleup_{J;\kappa}^{n,\eta}g\right|\right)\left|\bigtriangleup_{I;\kappa}^{n-1,\eta}f\left(x\right)\right|dx$$

$$\leq 2^{-r\kappa}2^{-\left|k\right|\min\{\kappa,N-2\kappa\}}\int_{\mathbb{R}^{n}}\sqrt{\sum_{(I,J)\in\mathcal{R}_{*}^{k,r}}\left(\int_{J_{\eta}}\left|\bigtriangleup_{J;\kappa}^{n,\eta}g\right|\right)^{2}}\sqrt{\sum_{(I,J)\in\mathcal{R}_{*}^{k,r}}\left|\bigtriangleup_{I;\kappa}^{n-1,\eta}f\left(x\right)\right|^{2}}dx$$

$$\leq 2^{-r\kappa}2^{-\left|k\right|\min\{\kappa,N-2\kappa\}}\left(\int_{\mathbb{R}^{n}}\left(\sum_{(I,J)\in\mathcal{R}_{*}^{k,r}}\left(\int_{J_{\eta}}\left|\bigtriangleup_{J;\kappa}^{n,\eta}g\right|\right)^{2}\right)^{\frac{p'}{2}}dx\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^{n}}\left(\sum_{(I,J)\in\mathcal{R}_{*}^{k,r}}\left|\bigtriangleup_{I;\kappa}^{n-1,\eta}f\left(x\right)\right|^{2}\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}$$

where the square function estimate (2.1) shows that

$$\left(\int_{\mathbb{R}^{n}}\left(\sum_{(I,J)\in\mathcal{R}_{*}^{k,r}}\left|\bigtriangleup_{I;\kappa}^{n-1,\eta}f\left(x\right)\right|^{2}\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\lesssim\left\|f\right\|_{L^{p}},$$

since for each $I \in \mathcal{G}$, there is at most one cube $J \in \mathcal{D}$ such that $(I,J) \in \mathcal{R}_*^{k,r}$. On the other hand, for each fixed $J \in \mathcal{D}$, the number of cubes $I \in \mathcal{G}$ such that $(I,J) \in \mathcal{R}_*^{k,r}$ is approximately $2^{r(n-1)}$, and so

$$\begin{aligned}
&\sum_{(I,J)\in\mathcal{R}_*^{k,r}} \left| \left\langle T \bigtriangleup_{I;\kappa}^{n-1,\eta} f, \bigtriangleup_{J;\kappa}^{n,\eta} g \right\rangle \right| \\
&\lesssim 2^{-r\kappa} 2^{-|k| \min\{\kappa, N-2\kappa\}} \left(\int_{\mathbb{R}^n} \left(\sum_{J\in\mathcal{D}} 2^{r(n-1)} \left(\int_{J_\eta} \left| \bigtriangleup_{J;\kappa}^{n,\eta} g \right| \right)^2 \right)^{\frac{p'}{2}} dx \right)^{\frac{1}{p'}} \|f\|_{L^p} \\
&\approx 2^{-r\left(\kappa - \frac{n-1}{2}\right)} 2^{-|k| \min\{\kappa, N-2\kappa\}} \|g\|_{L^{p'}} \|f\|_{L^p} ,
\end{aligned}$$

for $1 < p < \infty$ by the square function estimate (1.17) again.

7.2.1. *The enlarged form.* For $k \geq 0$ define

$$\mathcal{E}_*^{k,r} \equiv \left\{ (I,J) \in \mathcal{G}[U] \times \mathcal{D}_* : \ell(J) = 2^k,\ \ell(\pi_{\tan} J) = 2^r \ell(I),\ \text{and } I \subset C_{\text{pseudo}} 2^k \pi_{\tan} J \right\},$$

and define the *enlarged* form,

$$\mathsf{B}_{\text{enlarge}}(f,g) \equiv \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \sum_{(I,J)\in\mathcal{E}_*^{k,r}} \left\langle T \bigtriangleup_{I;\kappa}^{n-1,\eta} f, \bigtriangleup_{J;\kappa}^{n,\eta} g \right\rangle .$$

Then for each fixed $J \in \mathcal{D}$, the number of cubes $I \in \mathcal{G}$ such that $(I,J) \in \mathcal{E}_*^{k,r}$ is approximately $\frac{|2^k \pi_{\tan} J|}{|I|} = \frac{2^{k(n-1)} |\pi_{\tan} J|}{2^{-r(n-1)} |\pi_{\tan} J|} = 2^{(r+k)(n-1)}$, and so we have

$$\begin{aligned}
&\sum_{(I,J)\in\mathcal{R}_*^{k,r}} \left| \left\langle T \bigtriangleup_{I;\kappa}^{n-1,\eta} f, \bigtriangleup_{J;\kappa}^{n,\eta} g \right\rangle \right| \\
&\lesssim 2^{-r\kappa} 2^{-|k| \min\{\kappa, N-2\kappa\}} \left(\int_{\mathbb{R}^n} \left(\sum_{J\in\mathcal{D}} 2^{(r+k)(n-1)} \left(\int_{J_\eta} \left| \bigtriangleup_{J;\kappa}^{n,\eta} g \right| \right)^2 \right)^{\frac{p'}{2}} dx \right)^{\frac{1}{p'}} \|f\|_{L^p} \\
&\approx 2^{-r\left(\kappa - \frac{n-1}{2}\right)} 2^{-|k| \min\left\{\kappa - \frac{n-1}{2}, N-2\kappa - \frac{n-1}{2}\right\}} \|g\|_{L^{p'}} \|f\|_{L^p} ,
\end{aligned}$$

for $1 < p < \infty$ by the square function estimate (1.17) again.

7.3. **Wrapup.** Finally, taking $\kappa > \frac{n-1}{2}$, $N > 2\kappa$ and summing the above estimates over $r \in \mathbb{N}$ and $k \in \mathbb{Z}$, gives,

$$\left| \sum_{(I,J)\in\mathcal{R}_*} \left\langle T \bigtriangleup_{I;\kappa}^{n-1,\eta} f, \bigtriangleup_{J;\kappa}^{n,\eta} g \right\rangle \right| \lesssim \|f\|_{L^p} \|g\|_{L^{p'}} .$$

Combined with the reduction in the first subsection, we obtain the desired bound,

$$|\mathsf{B}_{\text{above}}(f,g)| \lesssim \|f\|_{L^p} \|g\|_{L^{p'}} , \qquad 1 < p < \infty,$$

in fact the stronger bound (7.1).

Remark 41. *The only restriction on p here is $1 < p < \infty$, and so the above form $\mathsf{B}_{\text{above}}(f,g)$ is bounded for all $1 < p < \infty$.*

8. Control of the upper disjoint and upper distal forms

The principle of stationary phase is not used for the disjoint or distal subforms, as the critical point of the phase now lies outside the support of the amplitude. When $k \geq 0$ we must introduce the radial integration by parts principle of decay to bound the subforms, while in the case $k \leq 0$, we must use the high order vanishing moments of $h_{J;\kappa}^{n,\eta}$. Just as in the case of the below form $\mathsf{B}_{\text{below}}$, combining the appropriate formulas, and staying the introduction of absolute values until the very end, will yield the desired inequalities. There is however a crucial difference between the cases $d \geq 0$ and $d < 0$ in the case of both disjoint subforms

$\mathsf{B}_{\mathrm{disjoint}}^{k,d,m}(f,g)$ and distal subforms $\mathsf{B}_{\mathrm{distal}}^{k,d}(f,g)$, and we will treat the upper and lower cases in separate subsections, as the resonant lower forms with $d<0$ require probability and interpolation techniques.

In fact, when $d\geq 0$, the standard principles of decay apply to give the required control. However, as d becomes increasingly negative, resonance begins to set in more strongly, and by the time $d=-m$, none of the standard principles of decay are any longer of use. Instead we must invoke classical methods of estimating L^2 and L^4 bounds, but using *probability* in order to obtain improved bounds for functions restricted to smooth Alpert pseudoprojections.

Recall that

$$\begin{aligned}
\mathsf{B}_{\mathrm{disjoint}}^{k,d,m}(f,g) &\equiv \sum_{(I,J)\in\mathcal{P}_m^{k,d}} \left\langle T\,\triangle_{I;\kappa}^{n-1,\eta} f,\triangle_{J;\kappa}^{n,\eta} g\right\rangle ,\\
\text{where } \mathcal{P}_m^{k,d} &\equiv \left\{(I,J)\in\mathcal{P}_m : \ell(J)=2^k, \text{ and } 2^d\leq \ell(I)^2\operatorname{dist}(0,J)\leq 2^{d+1}\right\},\\
\text{and } \mathcal{P}_m &\equiv \left\{(I,J)\in\mathcal{G}[U]\times\mathcal{D} : 2^{m+1}I\subset S \text{ and } \pi_{\tan}(J)\subset\Phi\left(2^{m+1}C_{\mathrm{pseudo}}I\right)\setminus\Phi\left(2^m C_{\mathrm{pseudo}}I\right)\right\},
\end{aligned}$$

and that the parameters (k,d,m) run over

$$k\in\mathbb{Z},\quad m\in\mathbb{N},\quad\text{and}\quad -\log_2\frac{1}{\ell(I)}\leq d<\infty .$$

We then decompose the disjoint form into upper and lower components determined by d nonnegative and negative respectively,

$$\begin{aligned}
\mathsf{B}_{\mathrm{disjoint}}(f,g) &= \mathsf{B}_{\mathrm{disjoint}}^{\mathrm{upper}}(f,g)+\mathsf{B}_{\mathrm{disjoint}}^{\mathrm{lower}}(f,g),\\
\mathsf{B}_{\mathrm{disjoint}}^{\mathrm{upper}}(f,g) &\equiv \sum_{m=1}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{d\geq 0}\mathsf{B}_{\mathrm{disjoint}}^{k,d,m}(f,g) \text{ and } \mathsf{B}_{\mathrm{disjoint}}^{\mathrm{lower}}(f,g)\equiv\sum_{m=1}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{d<0}\mathsf{B}_{\mathrm{disjoint}}^{k,d,m}(f,g).
\end{aligned}$$

For the distal form we write,

$$\begin{aligned}
\mathsf{B}_{\mathrm{distal}}^{k,d}(f,g) &\equiv \sum_{(I,J)\in\mathcal{X}^{k,d}} \left\langle T\,\triangle_{I;\kappa}^{n-1,\eta} f,\triangle_{J;\kappa}^{n,\eta} g\right\rangle ,\\
\text{where } \mathcal{X}^{k,d} &\equiv \left\{(I,J)\in\mathcal{X} : \ell(J)=2^k, \text{ and } 2^d\leq \ell(I)^2\operatorname{dist}(0,J)\leq 2^{d+1}\right\},\\
\text{and } \mathcal{X} &\equiv \left\{(I,J)\in\mathcal{G}[U]\times\mathcal{D} : 2^{m+1}I\subset S \text{ and } \pi_{\tan}(J)\cap\Phi(2U)=\emptyset\right\},
\end{aligned}$$

and decompose it into upper and lower subforms in the analogous way,

$$\begin{aligned}
\mathsf{B}_{\mathrm{distal}}(f,g) &= \mathsf{B}_{\mathrm{distal}}^{\mathrm{upper}}(f,g)+\mathsf{B}_{\mathrm{distal}}^{\mathrm{lower}}(f,g),\\
\mathsf{B}_{\mathrm{distal}}^{\mathrm{upper}}(f,g) &\equiv \sum_{k\in\mathbb{Z}}\sum_{d\geq 0}\mathsf{B}_{\mathrm{distal}}^{k,d}(f,g) \text{ and } \mathsf{B}_{\mathrm{distal}}^{\mathrm{lower}}(f,g)\equiv\sum_{k\in\mathbb{Z}}\sum_{d<0}\mathsf{B}_{\mathrm{distal}}^{k,d}(f,g).
\end{aligned}$$

8.1. **Upper disjoint subforms with $d\geq 0$.** When $k=0$, we obtain geometric gain simultaneously in $m\geq 1$ and $d\geq 0$ using the tangential integration by parts principle of decay. In order to handle arbitrary $k\in\mathbb{Z}$, we must include additional principles of decay combined with tangential integration by parts. For $k\geq 0$, we include radial integration by parts, and taking absolute values inside the integral at the very end, we will obtain below that,

$$\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim 2^{-kN_1}2^{-N_2(m+d)}\sqrt{|I|\,|J|}. \tag{8.1}$$

For $k\leq 0$, we include instead the moment vanishing properties of $h_{J;\kappa}^{n,\eta}$, and taking absolute values inside the integral at the very end, we will obtain below that,

$$\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim 2^{-|k|\kappa}2^{-N_2(m+d)}\sqrt{|I|\,|J|}. \tag{8.2}$$

With these estimates in hand, together with the square function arguments used repeatedly above, we obtain,

$$\left|\mathsf{B}_{\mathrm{disjoint}}^{k,d,m}(f,g)\right|\lesssim 2^{-\delta|k|}2^{-\delta(m+d)}\left(\int\left|\triangle_{I;\kappa}^{n-1,\eta}f\right|\right)\left(\int\left|\triangle_{J;\kappa}^{n,\eta}g\right|\right),\qquad\text{for } p\geq\frac{2n}{n-1},$$

for some $\delta > 0$ provided κ, N_1 and N_2 are chosen sufficiently large, and finally then,

$$\sum_{k\in\mathbb{Z}}\sum_{d\geq 0}\sum_{m=1}^{\infty}\left|\mathsf{B}_{\mathrm{disjoint}}^{k,d,m}\left(f,g\right)\right|\lesssim\left\|f\right\|_{L^p}\left\|g\right\|_{L^{p'}}\ ,\qquad \text{for } p\geq\frac{2n}{n-1}.$$

Here is a brief sketch of the two inner product estimates mentioned above, followed by the appropriate square function estimate.

8.1.1. *The case $k\geq 0, d\geq 0$.* Combining the radial integration by parts formula (4.7),

$$\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle=\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{e^{i\lambda\phi(x,y)}}{\phi\left(x,y\right)^{N_1}}\varphi_I^{\eta}\left(x\right)\partial_{\lambda}^{N_1}\widehat{\psi}_J^{\eta}\left(y,\lambda\right)dxdyd\lambda,$$

with the tangential integration by parts formula (4.19),

$$\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle=i^N\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}e^{i\lambda\phi(x,y)}\left\{\left(D_{\mathbf{v}}^x\frac{1}{\left(D_{\mathbf{v}}\Phi\right)\left(x\right)\cdot\Phi\left(y\right)}\right)^{N_2}\right\}\varphi_I^{\eta}\left(x\right)\widehat{\psi}_J^{\eta}\left(y,\lambda\right)dxdy\frac{d\lambda}{\lambda^{N_2}}.$$

gives

$$\begin{aligned}\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle&=i^N\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}e^{i\lambda\phi(x,y)}\left\{\left(D_{\mathbf{v}}^x\frac{1}{\left(D_{\mathbf{v}}\Phi\right)\left(x\right)\cdot\Phi\left(y\right)}\right)^{N_2}\right\}\varphi_I^{\eta}\left(x\right)\widehat{\psi}_J^{\eta}\left(y,\lambda\right)dxdy\frac{d\lambda}{\lambda^{N_2}}\\&=i^N\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{e^{i\lambda\phi(x,y)}}{\phi\left(x,y\right)^{N_1}}\left\{\left(D_{\mathbf{v}}^x\frac{1}{\left(D_{\mathbf{v}}\Phi\right)\left(x\right)\cdot\Phi\left(y\right)}\right)^{N_2}\right\}\varphi_I^{\eta}\left(x\right)\partial_{\lambda}^{N_1}\frac{\widehat{\psi}_J^{\eta}\left(y,\lambda\right)}{\lambda^{N_2}}dxdyd\lambda.\end{aligned}$$

Taking absolute values inside the integral, and using (4.8) together with $\min\left\{\frac{1}{\eta\ell(J)},\frac{1}{\lambda}\right\}\lesssim\frac{1}{\ell(J)}$, and (4.21), we obtain,

$$\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim 2^{-kN_1}2^{-N_2(m+d)}\sqrt{\left|I\right|\left|J\right|},\tag{8.3}$$

as required.

8.1.2. *The case $k\leq 0, d\geq 0$.* This time we use (4.19),

$$\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle=i^N\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}e^{i\lambda\phi(x,y)}\left\{\left(D_{\mathbf{v}}^x\frac{1}{\left(D_{\mathbf{v}}\Phi\right)\left(x\right)\cdot\Phi\left(y\right)}\right)^{N}\right\}\varphi_I^{\eta}\left(x\right)\widehat{\psi}_J^{\eta}\left(y,\lambda\right)dxdy\frac{d\lambda}{\lambda^{N}},$$

together with (4.13),

$$\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle=\int_{\mathbb{R}^{n-1}}e^{-i\Phi(x)\cdot c_J}h_{I;\kappa}^{n-1,\eta}\left(x\right)\left\{\int_{\mathbb{R}^n}R_{\kappa}\left(-i\Phi\left(x\right)\cdot\left(\xi-c_J\right)\right)h_{J;\kappa}^{n,\eta}\left(\xi\right)d\xi\right\}dx,$$

to obtain,

$$\begin{aligned}\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle&=i^N\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}e^{-i\lambda\phi(x,y)}\left\{\left(D_{\mathbf{v}}^x\frac{1}{\left(D_{\mathbf{v}}\Phi\right)\left(x\right)\cdot\Phi\left(y\right)}\right)^{N}\right\}\varphi_I^{\eta}\left(x\right)\widehat{\psi}_J^{\eta}\left(y,\lambda\right)dxdy\frac{d\lambda}{\lambda^{N}}\\&=(-i)^N\int_{\mathbb{R}^{n-1}}\left\{\int_{\mathbb{R}^n}e^{-i\Phi(x)\cdot\xi}\left\{\left(D_{\mathbf{v}}^x\frac{1}{\left(D_{\mathbf{v}}\Phi\right)\left(x\right)\cdot\frac{\xi}{|\xi|}}\right)^{N}\right\}\frac{h_{J;\kappa}^{n,\eta}\left(\xi\right)}{\left|\xi\right|^N}d\xi\right\}h_{I;\kappa}^{n-1,\eta}\left(x\right)dx\\&=(-i)^N\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^{n-1}}e^{-i\Phi(x)\cdot c_J}\left[\frac{1}{\left|\xi\right|^N}\left(D_{\mathbf{v}}^x\frac{1}{\left(D_{\mathbf{v}}\Phi\right)\left(x\right)\cdot\frac{\xi}{|\xi|}}\right)^{N}R_{\kappa}\left(-i\Phi\left(x\right)\cdot\left(\xi-c_J\right)\right)h_{I;\kappa}^{n-1,\eta}\left(x\right)\right]dx\right\}h_{J;\kappa}^{n,\eta}\left(\xi\right)d\xi,\end{aligned}$$

where in the second line above, we have reversed the change of variable in (3.6). Now from the estimates used in (4.21) and (4.14) we obtain,

$$\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim 2^{-|k|\kappa}2^{-N(m+d)}\sqrt{\left|I\right|\left|J\right|},$$

as required.

8.1.3. *The square function argument for $d \geq 0$.* We follow the square function argument used for the below form $\mathsf{B}_{\text{below}}^{k,d}(f,g)$ when $k \geq 0, d \leq 0$. The only difference is that we now accumulate a factor of a large power of 2^m depending on n and p, but this will be offset by gains from integration by parts in both parameters m and d - and this uses in a crucial way that $d \geq 0$. We begin by writing the sum over $(I,J) \in \mathcal{P}_m^{k,d}$ as,

$$\sum_{(I,J)\in\mathcal{P}_m^{k,d}} = \sum_{\substack{(I,J)\in\mathcal{G}[U]\times\mathcal{D}:\ 2^{m+1}I\subset U \text{ and } \pi_{\tan}(J)\subset\Phi\left(2^{m+1}CI\right)\setminus\Phi(2^mCI) \\ \ell(J)=2^k \text{ and } 2^d\leq\ell(I)^2\operatorname{dist}(0,J)\leq 2^{d+1}}},$$

and

$$\begin{aligned}
\left|\mathsf{B}_{\text{disjoint}}^{k,d,m}(f,g)\right| &= \left|\sum_{(I,J)\in\mathcal{P}_m^{k,d}} \left\langle T\bigtriangleup_{I;\kappa}^{n-1,\eta} f, \bigtriangleup_{J;\kappa}^{n,\eta} g\right\rangle\right| \leq \sum_{(I,J)\in\mathcal{P}_m^{k,d}} \left|\left\langle T\bigtriangleup_{I;\kappa}^{n-1,\eta} f, \bigtriangleup_{J;\kappa}^{n,\eta} g\right\rangle\right| \\
&\lesssim \sum_{(I,J)\in\mathcal{P}_m^{k,d}} 2^{-|k|\kappa}2^{-N_2(m+d)}\left(\int\left|\bigtriangleup_{I;\kappa}^{n-1,\eta}f\right|\right)\left(\int\left|\bigtriangleup_{J;\kappa}^{n,\eta}g\right|\right) \\
&= 2^{-|k|\kappa}2^{-N_2(m+d)}\int_{\mathbb{R}^n}\sum_{(I,J)\in\mathcal{P}_m^{k,d}}\left(\int_{\mathbb{R}^{n-1}}\left|\bigtriangleup_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)\mathbf{1}_J(\xi)\left|\bigtriangleup_{J;\kappa}^{n,\eta}g(\xi)\right|d\xi \\
&\lesssim 2^{-|k|\kappa}2^{-N_2(m+d)}\int_{\mathbb{R}^n}\sqrt{\sum_{(I,J)\in\mathcal{P}_m^{k,d}}\left(2^{m(n-1)}\int_{I_\eta}\left|\bigtriangleup_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)^2\mathbf{1}_J(\xi)}\sqrt{\sum_{(I,J)\in\mathcal{P}_m^{k,d}}2^{-m(n-1)}\left|\bigtriangleup_{J;\kappa}^{n,\eta}g(\xi)\right|^2}d\xi,
\end{aligned}$$

which gives

$$\begin{aligned}
\left|\mathsf{B}_{\text{disjoint}}^{k,d,m}(f,g)\right| &\lesssim 2^{-|k|\kappa}2^{-N_2(m+d)}\left(\int_{\mathbb{R}^n}\left(\sum_{(I,J)\in\mathcal{P}_m^{k,d}}\left(2^{m(n-1)}\int_{I_\eta}\left|\bigtriangleup_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)^2\mathbf{1}_J(\xi)\right)^{\frac{p}{2}}d\xi\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_{\mathbb{R}^n}\left(\sum_{(I,J)\in\mathcal{P}_m^{k,d}}2^{-m(n-1)}\left|\bigtriangleup_{J;\kappa}^{n,\eta}g(\xi)\right|^2\right)^{\frac{p'}{2}}d\xi\right)^{\frac{1}{p'}} \\
&\equiv 2^{-|k|\kappa}2^{-N_2(m+d)}\Gamma_1\Gamma_2\ .
\end{aligned}$$

We first consider Γ_2 which satisfies,

$$\Gamma_2^{p'} = \int_{\mathbb{R}^n}\left(\sum_{(I,J)\in\mathcal{P}_m^{k,d}}2^{-m(n-1)}\left|\bigtriangleup_{J;\kappa}^{n,\eta}g(\xi)\right|^2\right)^{\frac{p'}{2}}d\xi \lesssim \int_{\mathbb{R}^n}\left(\sum_{J\in\mathcal{D}}\left|\bigtriangleup_{J;\kappa}^{n,\eta}g(\xi)\right|^2\right)^{\frac{p'}{2}}d\xi \approx \|g\|_{L^{p'}}^{p'}\ ,$$

since for a fixed J with $\ell(J) = 2^k$, the number of cubes I such that

$$(I,J)\in\mathcal{P}_m^{k,d} = \left\{(I,J)\in\mathcal{G}[U]\times\mathcal{D}: \begin{array}{c} 2^{m+1}I\subset U \text{ and } \pi_{\tan}(J)\subset\Phi\left(2^{m+1}CI\right)\setminus\Phi\left(2^{m-1}CI\right) \\ \text{and } \ell(J)=2^k \text{ and } 2^d\leq\ell(I)^2\operatorname{dist}(0,J)\leq 2^{d+1}\end{array}\right\}$$

is roughly $2^{m(n-1)}$, and where the final approximation is the square function estimate (1.17).

Now we turn to Γ_1 for which we have the estimate,

$$\begin{aligned}
\Gamma_1^p &= \int_{\mathbb{R}^n}\left(\sum_{(I,J)\in\mathcal{P}_m^{k,d}} 2^{2m(n-1)}\left(\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)^2 \mathbf{1}_J(\xi)\right)^{\frac{p}{2}} d\xi \\
&= 2^{pm(n-1)}\int_{\mathbb{R}^n}\left(\sum_{J\in\mathcal{D}_k}\sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_m^{k,d}}\left(\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)^2 \mathbf{1}_J(\xi)\right)^{\frac{p}{2}} d\xi \\
&= 2^{pm(n-1)}\int_{\mathbb{R}^n}\sum_{J\in\mathcal{D}_k}\left(\sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_m^{k,d}}\left(\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)^2\right)^{\frac{p}{2}} \mathbf{1}_J(\xi)\, d\xi \\
&= 2^{pm(n-1)}2^{kn}\sum_{J\in\mathcal{D}_k}\left(\sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_m^{k,d}}\left(\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)^2\right)^{\frac{p}{2}}.
\end{aligned}$$

Now for each $J\in\mathcal{D}_k$, the number of cubes $I\in\mathcal{G}[U]$ with $(I,J)\in\mathcal{P}_m^{k,d}$ is approximately 2^{mn}, and so we compute that,

$$\begin{aligned}
\left(\sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_m^{k,d}}\left(\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)^2\right)^{\frac{p}{2}} &\lesssim \left(\sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_m^{k,d}} 1\right)^{\frac{p}{2}-1}\sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_m^{k,d}}\left(\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)^p \\
&\approx 2^{mn\left(\frac{p}{2}-1\right)}\sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_m^{k,d}}\left(\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)^p,
\end{aligned}$$

and hence that

$$\begin{aligned}
\Gamma_1^p &\lesssim 2^{pm(n-1)}2^{kn}\sum_{J\in\mathcal{D}_k}\left(\sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_m^{k,d}}\left(\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)^2\right)^{\frac{p}{2}} \\
&\lesssim 2^{pm(n-1)}2^{kn}\sum_{J\in\mathcal{D}_k}2^{mn\left(\frac{p}{2}-1\right)}\sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_m^{k,d}}\left(\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|dx\right)^p \\
&\lesssim 2^{m\left[p(n-1)+n\left(\frac{p}{2}-1\right)\right]}2^{kn}\sum_{J\in\mathcal{D}_k}\sum_{I\in\mathcal{G}[U]:\ (I,J)\in\mathcal{P}_m^{k,d}}|I|^{\frac{p}{2}}\left(\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|^2dx\right)^{\frac{p}{2}} \\
&\approx 2^{m\left[\frac{3}{2}pn-(p+n)\right]}2^{kn}\sum_{I\in\mathcal{G}[U]}\left(\sum_{J\in\mathcal{D}_k:\ (I,J)\in\mathcal{P}_m^{k,d}} 1\right)|I|^p\left(\frac{1}{|I_\eta|}\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f(x)\right|^2dx\right)^{\frac{p}{2}},
\end{aligned}$$

where by the extension of (6.6) to $m\geq 1$,

$$\sum_{J\in\mathcal{D}_k:\ (I,J)\in\mathcal{P}_m^{k,d}} 1 \approx 2^{m(n-1)}2^{-kn}\left|\mathcal{K}_d(I)\right| \approx 2^{m(n-1)}2^{-kn}2^{dn}\left(\frac{1}{|I|}\right)^{\frac{n+1}{n-1}}.$$

Thus we have

$$\begin{aligned}\Gamma_1^p &\lesssim 2^{m\left[\frac{3}{2}pn-(p+n)\right]}2^{kn}2^{m(n-1)}2^{-kn}2^{dn}\sum_{I\in\mathcal{G}[U]}\left(\frac{1}{|I|}\right)^{\frac{n+1}{n-1}}|I|^p\left(\frac{1}{|I_\eta|}\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}\right|^2\right)^{\frac{p}{2}}\\ &= 2^{m\left[\frac{3}{2}pn-(p+1)\right]}2^{dn}\sum_{I\in\mathcal{G}[U]}|I|^{p-\frac{n+1}{n-1}-1}\left(\frac{1}{|I_\eta|}\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f\right|^2\right)^{\frac{p}{2}}\mathbf{1}_I(x)\,dx\\ &\lesssim 2^{m\left[\frac{3}{2}pn-(p+1)\right]}2^{dn}\int_{\mathbb{R}^{n-1}}\sum_{I\in\mathcal{G}[U]}\left(\frac{1}{|I_\eta|}\int_{I_\eta}\left|\triangle_{I;\kappa}^{n-1,\eta}f\right|^2\mathbf{1}_I(x)\right)^{\frac{p}{2}}dx,\end{aligned}$$

if $p\geq\frac{2n}{n-1}$, and then using $p\geq 2$ and the Fefferman Stein vector valued inequality, we can continue with

$$\begin{aligned}\Gamma_1^p &\lesssim 2^{m\left[\frac{3}{2}pn-(p+1)\right]}2^{dn}\int_{\mathbb{R}^{n-1}}\left(\sum_{I\in\mathcal{G}[U]}\left(M\left|\triangle_{I;\kappa}^{n-1,\eta}f\right|^2\right)(x)\right)^{\frac{p}{2}}dx\\ &\lesssim 2^{m\left[\frac{3}{2}pn-(p+1)\right]}2^{dn}\int_{\mathbb{R}^{n-1}}\left(\sum_{I\in\mathcal{G}[U]}\left|\triangle_{I;\kappa}^{n-1,\eta}f\right|^2(x)\right)^{\frac{p}{2}}dx\lesssim 2^{m\left[\frac{3}{2}pn-(p+1)\right]}2^{dn}\left\|f\right\|_{L^p}^p\ .\end{aligned}$$

Altogether then we have

$$\begin{aligned}\left|\mathsf{B}_{\mathrm{disjoint}}^{k,d,m}(f,g)\right| &\lesssim 2^{-|k|\kappa}2^{-N_2(m+d)}\Gamma_1\Gamma_2\lesssim 2^{-|k|\kappa}2^{-N_2(m+d)}2^{m\left[\frac{3}{2}pn-(p+1)\right]}2^{dn}\left\|f\right\|_{L^p}\left\|g\right\|_{L^{p'}}\\ &= 2^{-|k|\kappa}2^{-\left(N_2-\frac{3}{2}pn+(p+1)\right)m}2^{-(N_2-n)d}2^{dn}\left\|f\right\|_{L^p}\left\|g\right\|_{L^{p'}}\leq 2^{-|k|\delta}2^{-\delta m}2^{-\delta d}\left\|f\right\|_{L^p}\left\|g\right\|_{L^{p'}}\ ,\end{aligned}$$

for $d\geq 0$ and $p\geq\frac{2n}{n-1}$, so

$$\sum_{k\in\mathbb{Z}}\sum_{d=0}^{\infty}\sum_{m=1}^{\infty}\left|\mathsf{B}_{\mathrm{disjoint}}^{k,d,m}(f,g)\right|\lesssim\sum_{k\in\mathbb{Z}}\sum_{d=0}^{\infty}\sum_{m=1}^{\infty}2^{-|k|\delta}2^{-\delta m}2^{-\delta d}\left\|f\right\|_{L^p}\left\|g\right\|_{L^{p'}}\lesssim\left\|f\right\|_{L^p}\left\|g\right\|_{L^{p'}}\ .$$

8.2. **Upper distal subforms with $d\geq 0$.** We can obtain similar estimates for the upper distal form, by treating this form as the sum over pairs (I,J) with J in the 'missing sector', i.e. by setting $m=s$ in the corresponding disjoint form estimates, as we now do. Indeed, recall that in (8.1) and (8.2) above we showed that

$$\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim 2^{-|k|\min\{N_1,\kappa\}}2^{-N_2(m+d)}\sqrt{|I|\,|J|},$$

for $(I,J)\in\mathcal{P}_m^{k,d}$, $k\in\mathbb{N}$ and $d\geq 0$. The same arguments, when applied to $(I,J)\in\mathcal{X}^{k,d}$, yield

$$\left|\left\langle Th_{I;\kappa}^{n-1,\eta},h_{J;\kappa}^{n,\eta}\right\rangle\right|\lesssim 2^{-|k|\min\{N_1,\kappa\}}2^{-N(s+d)}\sqrt{|I|\,|J|}\lesssim 2^{-|k|N_1}2^{-Nd}\sqrt{|I|\,|J|},$$

for $(I,J)\in\mathcal{X}^{k,d}$, $k\in\mathbb{N}$ and $d\geq 0$. Then the square function argument in the previous subsubsection applies to give

$$\sum_{k\in\mathbb{Z}}\sum_{d=0}^{\infty}\left|\mathsf{B}_{\mathrm{distal}}^{k,d}(f,g)\right|\lesssim\sum_{k\in\mathbb{Z}}\sum_{d=0}^{\infty}2^{-|k|\delta}2^{-\delta d}\left\|f\right\|_{L^p}\left\|g\right\|_{L^{p'}}\lesssim\left\|f\right\|_{L^p}\left\|g\right\|_{L^{p'}}\ ,$$

for some $\delta>0$.

8.3. **Wrapup.** If we define

$$\begin{aligned}\left|\mathsf{B}_{\mathrm{disjoint}}^{\mathrm{upper}}\right|(f,g) &\equiv \sum_{m=1}^{\infty}\sum_{(I,J)\in\mathcal{P}_m:\ \ell(I)^2\operatorname{dist}(0,J)\geq 1}\left|\left\langle T\triangle_{I;\kappa}^{n-1,\eta}f,\triangle_{J;\kappa}^{n,\eta}g\right\rangle\right|,\\ \left|\mathsf{B}_{\mathrm{distal}}^{\mathrm{upper}}\right|(f,g) &\equiv \sum_{(I,J)\in\mathcal{X}:\ \ell(I)^2\operatorname{dist}(0,J)\geq 1}\left|\left\langle T\triangle_{I;\kappa}^{n-1,\eta}f,\triangle_{J;\kappa}^{n,\eta}g\right\rangle\right|,\end{aligned}$$

in which the absolute values are taken inside the sums, we have proved both

$$\left|\mathsf{B}_{\mathrm{disjoint}}^{\mathrm{upper}}\right|(f,g)\lesssim\left\|f\right\|_{L^p}\left\|g\right\|_{L^{p'}}\ ,\qquad\text{for }p>\frac{2n}{n-1},\tag{8.4}$$

and

$$|\mathsf{B}_{\mathrm{distal}}^{\mathrm{upper}}|(f,g) \lesssim \|f\|_{L^p}\|g\|_{L^{p'}}, \qquad \text{for } p > \frac{2n}{n-1}. \tag{8.5}$$

9. Control of the lower disjoint and lower distal forms

Here we first bound the *lower* disjoint form

$$\mathsf{B}_{\mathrm{disjoint}}^{\mathrm{lower}}(f,g) \equiv \sum_{k\in\mathbb{Z}}\sum_{d<0}\sum_{m=1}^{\infty}\mathsf{B}_{\mathrm{disjoint}}^{k,d,m}(f,g),$$

which is the sum of all the disjoint subforms

$$\mathsf{B}_{\mathrm{disjoint}}^{k,d,m}(f,g) \equiv \sum_{(I,J)\in\mathcal{P}_m^{k,d}}\left\langle T\,\triangle_{I;\kappa}^{n-1,\eta}f, \triangle_{J;\kappa}^{n,\eta}g\right\rangle,$$

but taken only over $d<0$, which is equivalent to $\mathrm{dist}(0,J) \lesssim \frac{1}{\ell(I)^2}$. This restriction describes the 'lower' region of the disjoint form, and accounts for the terminology. At the end of this section we indicate how the same techniques can be used to control the lower distal form $\mathsf{B}_{\mathrm{distal}}^{\mathrm{lower}}(f,g)$.

9.1. Control of the lower disjoint form.

Note that for fixed $\xi\in\mathbb{R}^n$, the wavelength of the oscillation of the function $x\rightarrow e^{-i\Phi(x)\cdot\xi}$ is roughly $\frac{1}{|\xi|}\approx\frac{\ell(I)^2}{2^d}$, while the depth of the patch of the sphere $\Phi(I)$ in the direction toward ξ is roughly $\ell(I)\sin\theta\approx 2^m\ell(I)^2$. Thus we will have *oscillation* along the patch $\Phi(I)$ if and only if the wavelength $\frac{\ell(I)^2}{2^d}$ is less than the depth $2^m\ell(I)^2$, i.e. $m\gg|d|$, while we will have *smoothness* along the patch if and only if $m\ll|d|$.

On the other hand, for $\xi\in J$, the wavelength of the oscillation of the function $\xi\rightarrow e^{-i\Phi(x)\cdot\xi}$ is roughly $\frac{1}{\cos\measuredangle(\Phi(x),c_J)}\approx 1$ (unless the unit vectors $\frac{c_J}{|c_J|}$ and $\Phi(c_I)$ are nearly orthogonal), while the depth of the cube in the diretion of ξ is roughly $\ell(J)=2^k$. Thus we will have *oscillation* along the cube J if and only if the wavelength 1 is less than the depth 2^k, i.e. $k\gg 0$, while we will have *smoothness* along the cube if and only if $k\ll 0$.

Conclusion 42. *The most problematic case occurs when $d<0$ and both $m\approx|d|$ and $k\approx 0$.*

We begin by illustrating our approach to controlling resonance in the most problematic of the subcases in the next subsection, and it is here that we require the use of *probability* and an interpolation argument. In such instances where we need to use expectation over 'martingale transforms', we will also need to apply this expectation to *norms* rather than bilinear forms, which introduces some complications.

In order to handle cases with partial resonance in the subsequent subsection, we introduce a different decomposition of the disjoint form into resonant pipes that respects resonance when $d<0$, and then apply principles of decay along with probability and the interpolation argument to control these remaining subcases.

9.2. The extreme resonant case.

The most resonant of the disjoint subforms is $\mathsf{B}_{\mathrm{disjoint}}^{k,d,m}(f,g)=\mathsf{B}_{\mathrm{disjoint}}^{0,-m,m}(f,g)$ when $\ell(J)=1$ and $d=-m$. Fix $(I,J)\in\mathcal{P}_m^{0,-m}$ and let $J_{\max}^m[I]$ be any dyadic cube in $\mathcal{D}$ satisfying the following conditions,

$$\begin{aligned} \ell(J_{\max}^m[I]) &= \frac{1}{\ell(I)}, \\ \mathrm{dist}(0,J_{\max}^m[I]) &\approx \frac{2^{-m}}{\ell(I)^2}, \\ \pi_{\tan}J_{\max}^m[I] &\subset 2^{m+1}I\setminus 2^{m-1}I, \\ \ell(\pi_{\tan}J_{\max}^m[I]) &= 2^m\ell(I), \end{aligned} \tag{9.1}$$

where $\ell(\pi_{\tan}J_{\max}^m[I])$ denotes the diameter of the quasicube $\pi_{\tan}J_{\max}^m[I]$. If $\ell(I)=2^{-s}$ with $s\geq m$ (which follows from (9.1) and $\ell(\pi_{\tan}J_{\max}^m[I])\lesssim 1$), then we have

$$\ell(J_{\max}^m[I])=2^s,\quad \mathrm{dist}(0,J_{\max}^m[I])\approx 2^{2s-m},\quad \ell(\pi_{\tan}J_{\max}^m[I])=\frac{\ell(J_{\max}^m[I])}{\mathrm{dist}(0,J_{\max}^m[I])}=2^{m-s}.$$

At this point we note that the cubes $J_{\max}^{m}[I]$ are essentially the maximal dyadic cubes that fit inside the annular conic region given by (9.1), and hence there are roughly $\frac{\operatorname{dist}(0,J_{\max}^{m}[I])}{\ell(J_{\max}^{m}[I])}\approx\frac{2^{2s-m}}{2^{s}}\approx 2^{s-m}$ such cubes stacked away from the origin. We enumerate these cubes by $\{J_{\max}^{m,t}[I]\}_{t=1}^{c2^{s-m}}$ and let

$$J_{\max}^{m,*}[I]\equiv\bigcup_{t=1}^{c2^{s-m}}J_{\max}^{m,t}[I] \tag{9.2}$$

denote their union. Thus $J_{\max}^{m,*}[I]$ is a *quasirectangle* of 'length' roughly $\operatorname{dist}(0,J_{\max}^{m}[I])\approx 2^{2s-m}$, and 'width' roughly 2^{s} - we say 'quasi' because $J_{\max}^{m,*}[I]$ is a union of dyadic cubes $J_{\max}^{m,t}[I]$ staggered in the direction of the annular conic region. Note that there are at most C_n such quasirectangles $J_{\max}^{m,*}[I]$ associated to any given cube $I\in\mathcal{G}[S]$.

Remark 43. *Since quasirectangles do not respect resonance (which varies along the quasirectangle), they will not play a part in the proof going forward, but will instead be replaced by pipes in the next subsection.*

If $\phi\equiv\measuredangle\left(c_{J_{\max}^{m}[I]}-\Phi(c_I),\Phi(c_I)^{\perp}\right)$ is the angle between the vector $c_{J_{\max}^{m}[I]}-\Phi(c_I)$ and the unit vector $\Phi(c_I)$, and if $\theta\equiv\measuredangle\left(\frac{c_{J_{\max}^{m}[I]}}{\left|c_{J_{\max}^{m}[I]}\right|},\Phi(c_I)\right)$ is the angle between the unit vectors $\frac{c_{J_{\max}^{m}[I]}}{\left|c_{J_{\max}^{m}[I]}\right|}$ and $\Phi(c_I)$, then $\theta\approx 2^{m}\ell(I)$ and we have

$$\begin{aligned}\frac{\pi}{2}-\phi &= \measuredangle\left(c_{J_{\max}^{m}[I]}-\Phi(c_I),\Phi(c_I)\right)\\ &= \measuredangle\left(c_{J_{\max}^{m}[I]}-\frac{c_{J_{\max}^{m}[I]}}{\left|c_{J_{\max}^{m}[I]}\right|},\Phi(c_I)\right)+\measuredangle\left(c_{J_{\max}^{m}[I]}-\Phi(c_I),c_{J_{\max}^{m}[I]}-\frac{c_{J_{\max}^{m}[I]}}{\left|c_{J_{\max}^{m}[I]}\right|}\right)\\ &= \measuredangle\left(\frac{c_{J_{\max}^{m}[I]}}{\left|c_{J_{\max}^{m}[I]}\right|},\Phi(c_I)\right)+O\left(\frac{\left|\Phi(c_I)-\frac{c_{J_{\max}^{m}[I]}}{\left|c_{J_{\max}^{m}[I]}\right|}\right|}{\left|c_{J_{\max}^{m}[I]}-\Phi(c_I)\right|}\right)\approx 2^{m}\ell(I)+\frac{2^{m}\ell(I)}{\operatorname{dist}(0,J_{\max}^{m}[I])}\\ &= 2^{m}\ell(I)\left\{1+\frac{1}{\operatorname{dist}(0,J_{\max}^{m}[I])}\right\}\approx 2^{m}\ell(I)\left\{1+2^{m-2s}\right\}\approx 2^{m}\ell(I),\end{aligned} \tag{9.3}$$

since $s\geq m$. Thus it follows that there is neither oscillation nor smoothness of the inner product

$$\left\langle T\bigtriangleup_{I;\kappa}^{n-1,\eta}f,\bigtriangleup_{J;\kappa}^{n,\eta}g\right\rangle=\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^{n-1}}\left\langle f,h_{I;\kappa}^{n-1,\eta}\right\rangle h_{I;\kappa}^{n-1,\eta}(x)\,e^{i\Phi(x)\cdot\xi}dx\right\}\bigtriangleup_{J;\kappa}^{n,\eta}g(\xi)\,d\xi$$

in the integral over I in braces, since the 'tilted depth' of $\Phi(I)$ in the direction $\frac{\pi}{2}-\phi$ is given by

$$\text{tilted depth}\approx\ell(I)\cos\phi=\ell(I)\sin\left(\frac{\pi}{2}-\phi\right)\approx 2^{m}\ell(I)^{2},$$

and so

$$\text{wavelength}\approx\frac{1}{\operatorname{dist}(0,J_{\max}^{m}[I])}=2^{m}\ell(I)^{2}\approx\text{tilted depth}. \tag{9.4}$$

Of course there is neither oscillation nor smoothness in the integral over J either since $\ell(J)=1$ and the wavelength coming from the sphere is approximately $\ell(J)=1$ as well.

Then $(I,J)\in\mathcal{P}_{m}^{0,-m}$ *essentially* if and only if $J\subset J_{\max}^{m,*}[I]$ and $\ell(J)=1$. There are roughly $\frac{1}{\ell(I)^{n}}$ cubes $J\subset J_{\max}^{m,t}[I]$ of side length 1 for each $1\leq t\leq c2^{s-m}$, and we may restrict our attention to the cubes I having side length 2^{-s} with $s\geq m$, that are contained in a cube Q where

$$Q\subset S\text{ with }\ell(Q)\approx 2^{m-s},\text{ such that }J_{\max}^{m,*}[I]\approx J_{\max}^{m,*}[I']\text{ for all such cubes }I\subset Q. \tag{9.5}$$

We also then set

$$Q^{*}\equiv\bigcup_{I\subset Q}J_{\max}^{m,*}[I], \tag{9.6}$$

which is approximately equal to any of the $J_{\max}^{m,*}[I]$ taken individually, and thus Q^{*} is a quasirectangle of length roughly 2^{2s-m}, and width roughly 2^{s}. Thus we have defined cube / quasirectangle pairs (Q,Q^{*}) which we now analyze a bit further. Recall from (9.1) that $\ell(\pi_{\tan}Q^{*})\approx 2^{m}\ell(I)=2^{m-s}$.

We write

$$\mathsf{Q}_Q^s g \equiv \sum_{I\in \mathsf{Q}_Q^s} \triangle_{I;\kappa}^{n-1} g \text{ and } \mathsf{P}_{m,s}^{\eta,0,Q^*} g \equiv \sum_{J\subset Q^*:\ \ell(J)=1} \triangle_{J;\kappa}^{n,\eta} g, \tag{9.7}$$

and recalling that $\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}_Q^s\right)^{\spadesuit} = \left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}_Q^s\right)^{S_{\kappa,\eta}} = S_{\kappa,\eta}\mathcal{A}_{\mathbf{a}}\mathsf{Q}_Q^s\left(S_{\kappa,\eta}\right)^{-1}$ is the conjugation of $\mathcal{A}_{\mathbf{a}}\mathsf{Q}_Q^s$ by $S_{\kappa,\eta}$, we claim that

$$\begin{aligned} \mathbb{E}_{2^{\mathcal{D}}}^{\mu}\left|\sum_{m=1}^{\infty}\sum_{s=m}^{\infty}\sum_{Q}\left\langle T\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}_Q^s\right)^{\spadesuit} f, \mathsf{P}_{m,s}^{\eta,0,Q^*} g\right\rangle\right| &\leq \sum_{m=1}^{\infty}\sum_{s=m}^{\infty}\sum_{Q}\mathbb{E}_{2^{\mathcal{D}}}^{\mu}\left|\left\langle T\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}_Q^s\right)^{\spadesuit}, \mathsf{P}_{m,s}^{\eta,0,Q^*} g\right\rangle\right| \\ &\lesssim \left\|f\right\|_{L^p}\left\|g\right\|_{L^{p'}}, \qquad p\geq \frac{2n}{n-1}, \end{aligned} \tag{9.8}$$

where we recall that the parameters k and d are fixed at $k=0$ and $d=-m$. It is here in (9.8) that our argument requires averaging over all involutive smooth Alpert multipliers on the left hand side of the inequality. Note that we have replaced the large projection Q_S with the smaller projections Q_Q^s for $Q\subset S$.

9.2.1. *The interpolation argument.* In order to illustrate the probabilistic methods in a relatively simple situation, we first prove (9.8) when the sum is taken only over $s=m\in\mathbb{N}$, so that both Q and Q^* reduce to cubes of side length roughly 1. Thus there are only a bounded number of such cube / cube pairs (Q,Q^*), which for convenience we treat as a single pair (Q_0,Q_0^*). We claim,

$$\mathbb{E}_{2^{\mathcal{G}}}^{\mu}\left|\sum_{m=1}^{\infty}\left\langle T\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}_{Q_0}^m\right)^{\spadesuit} f, \mathsf{P}_{m,m}^{\eta,0,Q_0^*} g\right\rangle\right| \lesssim \left\|f\right\|_{L^p}\left\|g\right\|_{L^{p'}}, \qquad p>\frac{2n}{n-1}. \tag{9.9}$$

We note that the expectation $\mathbb{E}_{2^{\mathcal{G}}}^{\mu}$ will circumvent some of the geometric L^4 arguments that go back to Fefferman [Fef] (see also [Bou], [Gut] and [Tao4]). Recall that we are in the case $d=-m$, and that

$$\mathsf{Q}_{Q_0}^m g = \sum_{I\subset Q_0:\ \ell(I)=2^{-m}} \triangle_{I;\kappa}^{n-1} g \text{ and } \mathsf{P}_{m,m}^{\eta,0,Q_0^*} g \equiv \sum_{J\subset Q_0^*:\ \ell(J)=1} \triangle_{J;\kappa}^{n,\eta} g,$$

where Q_0 is a cube in $\mathbb{R}^{n-1}$ centered at the origin with side length approximately 1, and Q_0^* is a cube in $\mathbb{R}^n$ at distance 2^m from the origin with side length approximately 2^m, and such that $\operatorname{dist}\left(Q_0,\pi_{\tan}Q_0^*\right)\approx 1$. We will again use $\widehat{\varphi}$ to denote the Fourier transform of φ. Thus we must estimate the average of the moduli of the inner products,

$$\begin{aligned} \left\langle T\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}_{Q_0}^m\right)^{\spadesuit} f, \mathsf{P}_{m,m}^{\eta,0,Q_0^*} g\right\rangle &= \left\langle T\sum_{I\in\mathcal{G}_m[Q_0]} a_I\,\triangle_{I;\kappa}^{n-1,\eta} f, \sum_{J\subset Q_0^*:\ \ell(J)=1}\triangle_{J;\kappa}^{n,\eta} g\right\rangle \\ &= \sum_{I\in\mathcal{G}_m[Q_0]}\ \sum_{J\subset Q_0^*:\ \ell(J)=1}\int_S\int_{\mathbb{R}^n} e^{-i\Phi(x)\cdot\xi} a_I\,\triangle_{I;\kappa}^{n-1,\eta} f(x)\,\triangle_{J;\kappa}^{n,\eta} g(\xi)\,dxd\xi \\ &= \int_{\mathbb{R}^n}\left\{\int e^{-iz\cdot\xi}\sum_{I\in\mathcal{G}_m[Q_0]} a_I\,\triangle_{I;\kappa}^{n-1,\eta} f\left(\Phi^{-1}(z)\right)\partial\Phi^{-1}(z)\,dz\right\}\sum_{J\subset Q_0^*:\ \ell(J)=1}\triangle_{J;\kappa}^{n,\eta} g(\xi)\,d\xi \\ &\equiv \int_{\mathbb{R}^n}\widehat{f_{\mathbf{a},\Phi}}(\xi)\,g_m(\xi)\,d\xi, \end{aligned} \tag{9.10}$$

where $\widehat{f_{\mathbf{a},\Phi}}$ denotes the Fourier transform of $f_{\mathbf{a},\Phi}$ as in Subsection 5, and

$$\begin{aligned} g_m(\xi) &\equiv \sum_{J\subset Q_0^*:\ \ell(J)=1}\triangle_{J;\kappa}^{n,\eta} g(\xi) = \mathsf{P}_{m,m}^{\eta,0,Q_0^*} g(\xi), \\ f_{\mathbf{a},\Phi}(z) &\equiv \left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}_{Q_0}^m\right)^{\spadesuit} f\left(\Phi^{-1}(z)\right)\partial\Phi^{-1}(z) = \sum_{I\in\mathcal{G}_m[Q_0]} a_I\,\triangle_{I;\kappa}^{n-1,\eta} f\left(\Phi^{-1}(z)\right)\partial\Phi^{-1}(z) \\ &= \sum_{I\in\mathcal{G}_m[Q_0]} a_I\left\langle f, h_{I;\kappa}^{n-1,\eta}\right\rangle h_{I;\kappa}^{n-1,\eta}\left(\Phi^{-1}(z)\right)\partial\Phi^{-1}(z) \equiv \sum_{I\in\mathcal{G}_m[Q_0]} f_{\mathbf{a},\Phi}^I(z), \end{aligned}$$

and where the spherical measure $f_{\mathbf{a},\Phi}^I$ has mass roughly $\left|\widehat{f}(I)\right|2^{-m(n-1)}$ and is supported in $\mathbb{S}^{n-1}$.

The bound (9.9) now follows immediately from Hölder's inequality and Proposition 32, upon noting that Q_S^s in Proposition 32 is the projection $\mathsf{Q}_{Q_0}^m$ here. Indeed, from Proposition 32 we have

$$\sum_{m=1}^{\infty} \mathbb{E}_{2\mathcal{G}}^{\mu} \left\| T\left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_{Q_0}^m\right)^{\spadesuit} f \right\|_{L^p\left(|\varphi_m|^4\right)} \lesssim \sum_{m=1}^{\infty} 2^{-m\varepsilon_{n,p}} \|f\|_{L^p\left(|\varphi_m|^4\right)}$$

and then in particular,

$$\begin{aligned} &\mathbb{E}_{2\mathcal{G}}^{\mu} \left| \sum_{m=1}^{\infty} \left\langle T\left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_{Q_0}^m\right)^{\spadesuit} f, \mathsf{P}_{m,m}^{\eta,0,Q_0^*} g \right\rangle \right| \leq \sum_{m=1}^{\infty} \mathbb{E}_{2\mathcal{G}}^{\mu} \left\| T\left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_{Q_0}^m\right)^{\spadesuit} f \right\|_{L^p\left(|\varphi_m|^4\right)} \left\| \mathsf{P}_{m,m}^{\eta,0,Q_0^*} g \right\|_{L^{p'}\left(|\varphi_m|^4\right)} \\ \leq &\sum_{m=1}^{\infty} 2^{-m\varepsilon_{n,p}} \|f\|_{L^p\left(|\varphi_m|^4\right)} \|g\|_{L^{p'}\left(|\varphi_m|^4\right)} \lesssim \|f\|_{L^p} \|g\|_{L^{p'}} \ , \qquad \text{where } \varepsilon_{n,p} > 0 \text{ for } p > \frac{2n}{n-1}, m \in \mathbb{N}. \end{aligned}$$

But we can in fact obtain more. Define the smooth Alpert pseudoprojection

$$\mathsf{P}_{m,m}^{\eta,Q_0^*} g \equiv \sum_{k\in\mathbb{Z}} \sum_{J\subset Q_0^*:\ \ell(J)=2^k} \triangle_{J;\kappa}^{n,\eta} g, \tag{9.11}$$

where of course the restriction $J \subset Q_0^*$ means that $k \leq m$ in the sum above (contrast this with the restriction to $k = 0$ in $\mathsf{P}_{m,m}^{\eta,0,Q_0^*} g$). Then we have the stronger inequality in which the sum over k is included,

$$\begin{aligned} &\mathbb{E}_{2\mathcal{G}}^{\mu} \left| \sum_{m=1}^{\infty} \left\langle T_S\left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_{Q_0}^m\right)^{\spadesuit} f, \mathsf{P}_{m,m}^{\eta,Q_0^*} g \right\rangle \right| \leq \sum_{m=1}^{\infty} \mathbb{E}_{2\mathcal{G}}^{\mu} \left\| T_S\left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_{Q_0}^m\right)^{\spadesuit} f \right\|_{L^p} \left\| \mathsf{P}_{m,m}^{\eta,Q_0^*} g \right\|_{L^{p'}} \\ \leq &\sum_{m=1}^{\infty} 2^{-m\varepsilon_{p,n}} \left\| S_{\kappa,\eta} \mathcal{A}_{\mathbf{a}} \mathsf{Q}_{Q_0}^m \left(S_{\kappa,\eta}\right)^{-1} f \right\|_{L^p} \left\| \mathsf{P}_{m,m}^{\eta,Q_0^*} g \right\|_{L^{p'}} \lesssim \|f\|_{L^p} \|g\|_{L^{p'}} \ , \qquad p > \frac{2n}{n-1}, m \in \mathbb{N}. \end{aligned} \tag{9.12}$$

Remark 44. *There is no direct use here of square function estimates to add in the parameter m. Instead, we use expectation, geometric decay, and the boundedness of connected smooth Alpert pseudoprojections on L^p - a pseudoprojection is connected if the cubes are summed over a connected set in the grid. This feature will persist in summing over the additional parameters s and d below.*

9.3. **The resonant pipe decomposition.** In order to complete the proof of the main inequality (9.8), we will abandon the decomposition into cones parameterized by m, and distances parameterized by d, since this decomposition does not respect resonance in the inner products. Instead, we will decompose the lower disjoint form,

$$\mathsf{B}_{\text{disjoint}}^{\text{lower}}(f,g) = \sum_{k\in\mathbb{Z}} \sum_{d<0} \sum_{m=1}^{\infty} \mathsf{B}_{\text{disjoint}}^{k,d,m}(f,g) = \sum_{I\in\mathcal{G}_s[S]} \sum_{k\in\mathbb{Z}} \sum_{d<0} \sum_{m=1}^{\infty} \sum_{(I,J)\in\mathcal{P}_m^{k,d}} \left\langle T \triangle_{I;\kappa}^{\eta} f, \triangle_{J;\kappa}^{\eta} g \right\rangle,$$

into 'truncated pipes' $P_{s,w}^I$, instead of the quasirectangles $J_{\max}^{m,*}[I]$ introduced in (9.2) above, using *new* parameters w, r in place of m, d. The advantage of this new decomposition into pipes is that it does indeed respect resonance.

Fix $s \in \mathbb{N}$ and consider a cube $I \in G_s[U]$. Let $\mathbf{u}_n^I$ be the unit outward normal to the sphere at the point $\Phi(c_I)$, and let $\left(\mathbf{u}^I\right)' = \left\{\mathbf{u}_1^I, ..., \mathbf{u}_{n-1}^I\right\}$ be an orthonormal basis for the space $\left(\mathbf{u}_n^I\right)^{\perp}$ perpendicular to $\mathbf{u}_n^I$. We will use the coordinate system $\left\{\left(\mathbf{u}^I\right)', \mathbf{u}_n^I\right\}$ in $\mathbb{R}^n$ in connection with the cube $I \in G_s[U]$, so that as we vary $I \in G_s[U]$ the coordinate systems $\left\{\left(\mathbf{u}^I\right)', \mathbf{u}_n^I\right\}$ rotate (Span $\left\{\mathbf{u}_n^I\right\}$ and Span $\left(\mathbf{u}^I\right)'$ are determined canonically under rotation, but not the individual basis vectors $\mathbf{u}_1^I, ..., \mathbf{u}_{n-1}^I$).

For convenience in notation, we momentarily suppose without loss of generality that $I = I_0 \in G_s[U]$ is centered at the origin in S, and consequently we can take $\left\{\mathbf{u}_1^I, ..., \mathbf{u}_{n-1}^I, \mathbf{u}_n^I\right\}$ to be the standard orthonormal basis $\{\mathbf{e}_1, ..., \mathbf{e}_{n-1}, \mathbf{e}_n\}$ in $\mathbb{R}^n$, and $\xi = (\xi_1, ..., \xi_n) = (\xi', \xi_n) \in \mathbb{R}^n$ is the usual representation of a point ξ in $\mathbb{R}^n$. Then the pairs $(I_0, J) \in \mathcal{G}[U] \times \mathcal{D}$ for which we have resonance on both sides of the inner product, are precisely those satisfying $\ell(J) \approx 1$ and,

$$\begin{aligned}
(9.13) \qquad \frac{1}{\operatorname{dist}(0,J)} &\approx \text{tilted depth} \approx 2^{-s}\sin\theta, \\
\text{i.e. } |\xi| &\approx \frac{2^s}{\sin\theta} = 2^s\frac{|\xi|}{|\xi'|}, \qquad \text{for } \xi \in J, \\
\text{i.e. } 2^{s-1} &\leq |\xi'| \leq 2^{s+1}, \qquad \text{for } \xi \in J,
\end{aligned}$$

where θ is the angle ξ makes with the positive ξ_n-axis. Thus the union $P_s^{I_0}$ of the $J's$ satisfying $\ell(J) \approx 1$ and (9.13) is essentially the difference of two tubes, namely the 2^{s+1}-tube and the 2^{s-1} tube that are oriented vertically with length 2^{2s} and width 2^s, and centered on say the plane $\xi_n = 0$. We refer to $P_s^{I_0}$ as the resonant 2^s-pipe for I_0. In terms of the projection $\pi_{\Phi(c_{I_0})^\perp}$ of $\mathbb{R}^n$ onto the horizontal plane perpendicular to $\Phi(c_{I_0})$, we have

$$P_s^{I_0} \approx \left\{\xi \in \mathbb{R}^n : \operatorname{dist}\left(c_{I_0}, \pi_{\Phi(c_{I_0})^\perp}\xi\right) \approx 2^s\right\},$$

since $|\xi'| \approx \operatorname{dist}\left(c_{I_0}, \pi_{(c_{I_0})^\perp}\xi\right)$.

We also consider the *truncated* pipes

$$P_{s,w}^{I_0} \equiv P_s^{I_0} \cap L_w^{I_0}, \qquad 1 \leq w \leq s.$$

that are given as the intersection of the pipe $P_s^{I_0}$ and the horizontal slab $L_w^{I_0} \equiv \left\{\xi \in \mathbb{R}^n : 2^{2s-w-1} < \xi_n \leq 2^{2s-w}\right\}$ that is distance 2^{2s-w-1} above the plane $\xi_n = 0$ and has height roughly 2^{2s-w}.

We now extend this notion of pipes to all $I \in \mathcal{G}_s[U]$.

Definition 45. *For $I \in \mathcal{G}_s[S]$ and $0 \leq w \leq s$, define the truncated pipe $P_{s,w}^I$ to be the rotation of the pipe $P_{s,w}^{I_0}$ by any rotation R that takes $\Phi(c_{I_0})$ to $\Phi(c_I)$, i.e.*

$$P_{s,w}^I \equiv RP_{s,w}^{I_0} \approx \left\{\xi \in \mathbb{R}^n : \operatorname{dist}\left(c_{I_0}, \pi_{\Phi(c_I)^\perp}\xi\right) \approx 2^s\right\},$$

where $\pi_{\Phi(c_I)^\perp} = \pi_{R\Phi(c_{I_0})^\perp}$.

Note that if $|\xi'| \gg 2^s$ then $e^{-i\Phi(x)\cdot\xi}$ oscillates at least $\frac{|\xi'|}{2^s}$ times along the span of $\Phi(I)$, so that integration by parts is effective, while if $|\xi'| \ll 2^s$ then $e^{-i\Phi(x)\cdot\xi}$ varies by at most $\frac{|\xi'|}{2^s}$ along the span of $\Phi(I)$, so that the vanishing moment properties of $h_{I;\kappa}^\eta$ are effective.

Definition 46. *For $r > 0$ define the n-dimensional annulus $A_n(0,r)$ by*

$$A_n(0,r) \equiv B_n(0,r) \setminus B_n\left(0, \frac{r}{2}\right),$$

which we sometimes denote by simply $A(0,r)$. Define the upper quarter *annulus $A_+(0,r)$ by*

$$A_+(0,r) \equiv \left\{\xi \in A(0,r) : \xi_n \geq \frac{r}{4}\right\}.$$

Finally, we note that the upper quarter annulus $A_+\left(0, 2^{2s-w}\right)$ is essentially the union of the truncated pipes $P_{s,w}^I = P_s^I \cap L_w^I$ for $I \in \mathcal{G}_s[U]$, i.e. $A_+\left(0, 2^{2s-w}\right) \approx \bigcup_{I \in \mathcal{G}_s[U]} P_{s,w}^I$, and that the overlap of the truncated pipes $P_{s,w}^I$ is essentially $2^{w(n-1)}$, i.e.

$$\mathbf{1}_{A_+(0,2^{2s-w})}(\xi) \lesssim \frac{1}{2^{w(n-1)}} \sum_{I \in \mathcal{G}_s[U]} \mathbf{1}_{P_{s,w}^I}(\xi) \lesssim \mathbf{1}_{CA_+(0,2^{2s-w+c})}(\xi).$$

To complete control of the disjoint form in the case $d < 0$, it suffices to prove the following lemma. We will later establish average control of L^p norms instead of inner products, something that is needed to complete the proof of Theorem 4.

Lemma 47. *Suppose $s\in\mathbb{N}$ and $0\leq w\leq s$. Then*

$$\mathbb{E}^{\mu}_{2^{\mathcal{G}_s[U]}}\left|\left\langle T_S\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}^s_U\right)^{\spadesuit}f,\mathsf{P}^{\eta}_{A_+(0,2^{2s-w})}g\right\rangle\right|\lesssim 2^{-\varepsilon_{n,p}s}\left\|f\right\|_{L^p}\left\|g\right\|_{L^{p'}}\ ,\qquad \text{for } p>\frac{2n}{n-1},$$

where the implied constant is independent *of s and w.*

To prove the lemma, fix $0\leq w\leq s$ and $\mathbf{a}\in 2^{\mathcal{G}_s[S]}$, and consider the positive expression,

$$Z^{\mathbf{a}}_{s,w}\equiv\left|\sum_{I\in\mathcal{G}_s[U]}\sum_{J\subset P^I_{s,w}}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^{n-1}}e^{-i\Phi(x)\cdot\xi}\left(\mathcal{A}_{\mathbf{a}}\bigtriangleup^{n-1}_{I;\kappa}\mathsf{Q}^s_S\right)^{\spadesuit}f(x)\,dx\right\}\bigtriangleup^{n,\eta}_{J;\kappa}g(\xi)\,d\xi\right|, \tag{9.14}$$

which includes only the portion of the smooth pseudoprojection $\mathsf{P}^{\eta}_{A\left(0,2^{2(s-w)}\right)}g$ given by $\mathsf{P}^{\eta}_{P^I_{s,w}}g$. We begin by establishing control of $Z^{\mathbf{a}}_{s,w}$, and then control the sums over cubes J in expanding geometric annuli away from the truncated pipes $P^I_{s,w}$, by applying decay principles to obtain geometric decay factors. Finally we apply the arguments used to bound $Z^{\mathbf{a}}_{s,w}$ to each of these collections of annuli, and then sum up the annuli to cover all of the upper quarter annulus $A_+\left(0,2^{2s-w}\right)$, which completes the proof of the lemma.

Definition 48. *Define the* expanded *truncated pipes*

$$P^{I_0}_{s,w}[r]=\left\{\xi\in\mathbb{R}^n:\delta_r\xi\in P^{I_0}_{s,w}\right\},$$

where $\delta_r\xi=\left(\frac{\xi'}{2^r},\frac{\xi_n}{C_n}\right)$ is a nonisotropic dilation for $r\in\mathbb{Z}$ and C_n is chosen sufficiently large. Thus $P^{I_0}_{s,w}[r]$ is a truncated pipe of height roughly C_n2^{2s} and width roughly 2^{s+r} centered at a point horizontally located away from that of $P^{I_0}_{s,w}$. Then define the rotated expanded truncated pipes $P^I_{s,w}[r]$ for $I\in\mathcal{G}_s[S]$, by $P^I_{s,w}[r]\equiv RP^{I_0}_{s,w}[r]$ for any rotation R in $\mathbb{R}^n$ that takes c_{I_0} to c_I.

Note that if C_n is chosen sufficiently large in the definition of $P^{I_0}_{s,w}(r)$, then for every $I\in\mathcal{G}_s[U]$, the upper quarter annulus $A_+\left(0,2^{2s-w}\right)$ is contained in the union of the tube $T^I_{s,w}$, which we recall is the convex hull of the truncated pipe $P^I_{s,w}$, and the expanded truncated pipes $P^I_{s,w}[r]$ for $r\leq 2s-w$, i.e.

$$A_+\left(0,2^{2s-w}\right)\subset T^I_{s,w}\cup\left(\bigcup_{r=1}^{2s-w}P^I_{s,w}[r]\right),\qquad\text{for all } I\in\mathcal{G}_s[S]. \tag{9.15}$$

We will need to choose C_n even larger in Subsubsection 9.4 below.

Definition 49. *For $\mathbf{a}\in 2^{\mathcal{G}_s[S]}$ and $r\geq 0$, define*

$$Z^{\mathbf{a}}_{s,w}[r]\equiv\left|\sum_{I\in\mathcal{G}_s[U]}\sum_{J\subset P^I_{s,w}[r]}\int_{\mathbb{R}^n}\left\{\int e^{-i\Phi(x)\cdot\xi}\left(\mathcal{A}_{\mathbf{a}}\bigtriangleup^{n-1}_{I;\kappa}\right)^{\spadesuit}f(x)\,dx\right\}\bigtriangleup^{n,\eta}_{J;\kappa}g(\xi)\,d\xi\right|. \tag{9.16}$$

We will now control the average of this sum of inner products, as well as the stronger average norm estimates, see (9.18) below. First, we consider the two extreme cases $w=0$ and $w=s$, which are easily handled by two different techniques. Then we combine these two proofs to give a single argument for the general case.

Definition 50. *We define*

$$\mathcal{R}^{k,w}_s(r)\equiv\left\{(I,J)\in\mathcal{G}_s[U]\times\mathcal{D}_k:J\subset P^I_{s,w}[r]\right\}$$

to be the set of pairs $(I,J)\in\mathcal{G}[U]\times\mathcal{D}$ with $\ell(I)=2^{-s}$, $\ell(J)=2^k$ and $J\subset P^I_{s,w}(r)$. When $r=0$ we write simply

$$\mathcal{R}^{k,w}_s=\mathcal{R}^{k,w}_s(0).$$

For symmetry of notation, we also introduce tubes $\widehat{I_0}[w]$ that are essentially the same as the tubes $T^I_{s,w}$. For $I\in\mathcal{G}_s[U]$ and $0\leq w\leq s$, define

$$\widehat{I_0}[w]\equiv\left[-2^s,2^s\right]^{n-1}\times\left[2^{2s-w-1},2^{2s-w}\right]\approx T^{I_0}_{s,w},$$

and extend this definition to $\widehat{I}[w]$ by rotation , so that $\widehat{I}[w]\approx T^I_{s,w}$ and $\widehat{I}[0]\approx\widehat{I}$.

9.3.1. *The case $w=0$ (Direct Argument):* In the case $w=0$, we first consider $Z_{s,0}^{\mathbf{a}}$ with the sequence $\mathbf{a}=\mathbf{1}$ of all $1's$, since the arguments in this subsubsection take absolute values inside anyways, and do not use probability. The bound for the subform

$$Z_{s,0}^{\mathbf{1}}=\left|\sum_{s=1}^{\infty}\sum_{I\in\mathcal{G}_s[U]}\sum_{J\in\mathcal{D}:\ J\subset\widehat{I}}\left\langle T\bigtriangleup_{I;\kappa}^{n-1,\eta}f,\bigtriangleup_{J;\kappa}^{n,\eta}g\right\rangle\right|$$

applies more generally to indicators $\mathbf{1}_I$ applied to f, in place of smooth Alpert pseudoprojections $\bigtriangleup_{I;\kappa}^{n-1,\eta}$ applied to f, and to $\mathbf{1}_{\widehat{I}}$ in place of $\sum_{J\in\mathcal{D}:\ J\subset\widehat{I}}\bigtriangleup_{J;\kappa}^{n,\eta}$. To see this, we first note that

$$\begin{aligned}\left\|T\mathbf{1}_If\right\|_{L^p(\widehat{I})} &= \left(\int_{\widehat{I}}\left|\int_I e^{-i\Phi(x)\cdot\xi}f(x)\,dx\right|^p d\xi\right)^{\frac{1}{p}}\leq\left|\widehat{I}\right|^{\frac{1}{p}}|I|^{\frac{1}{p'}}\left(\int_I|f(x)|^p\,dx\right)^{\frac{1}{p}}\\ &= 2^{s\frac{n+1}{p}}2^{-s\frac{n-1}{p'}}\left\|\mathbf{1}_If\right\|_{L^p(\mathbb{R}^{n-1})}=2^{-s\varepsilon_{p,n}}\left\|\mathbf{1}_If\right\|_{L^p(\mathbb{R}^{n-1})}\ ,\end{aligned}$$

where

$$\varepsilon_{p,n}\equiv\frac{n-1}{p'}-\frac{n+1}{p}=\frac{n-1}{p}\left(p-1-\frac{n+1}{n-1}\right)=\frac{n-1}{p}\left(p-\frac{2n}{n-1}\right).$$

Then with s fixed, we continue with

$$\begin{aligned}&\sum_{I\in\mathcal{G}_s[U]}\left|\left\langle T\mathbf{1}_If,\mathbf{1}_{\widehat{I}}g\right\rangle\right|\leq\sum_{I\in\mathcal{G}_s[U]}\left\|T\mathbf{1}_If\right\|_{L^p(\widehat{I})}\left\|g\right\|_{L^{p'}(\widehat{I})}\leq\left(\sum_{I\in\mathcal{G}_s[U]}\left\|T\mathbf{1}_If\right\|_{L^p(\widehat{I})}^p\right)^{\frac{1}{p}}\left(\sum_{I\in\mathcal{G}_s[U]}\left\|g\right\|_{L^{p'}(\widehat{I})}^{p'}\right)^{\frac{1}{p'}}\\ \lesssim\ &\left(\sum_{I\in\mathcal{G}_s[U]}2^{-sp\varepsilon_{p,n}}\left\|\mathbf{1}_If\right\|_{L^p(\mathbb{R}^{n-1})}^p\right)^{\frac{1}{p}}\left\|g\right\|_{L^{p'}\left(\cup_{I\in\mathcal{G}_s[U]}\widehat{I}\right)}\leq2^{-s\varepsilon_{p,n}}\left\|f\right\|_{L^p(\mathbb{R}^{n-1})}\left\|g\right\|_{L^{p'}(\mathbb{R}^n)}\ ,\end{aligned}$$

and finally we sum over $s\in\mathbb{N}$ to obtain

$$\left|\sum_{s=1}^{\infty}\sum_{I\in\mathcal{G}_s[U]}\left\langle T\mathbf{1}_If,\mathbf{1}_{\widehat{I}}g\right\rangle\right|\leq\sum_{s=1}^{\infty}\sum_{I\in\mathcal{G}_s[U]}\left|\left\langle T\mathbf{1}_If,\mathbf{1}_{\widehat{I}}g\right\rangle\right|\leq C_{s,n}\left\|f\right\|_{L^p(\mathbb{R}^{n-1})}\left\|g\right\|_{L^{p'}(\mathbb{R}^n)}\ ,$$

where

$$C_{s,n}\equiv\sum_{s=1}^{\infty}2^{-\varepsilon_{p,n}s}<\infty\text{ for }p>\frac{2n}{n-1}.$$

Corollary 51. *If we enlarge the cubes I by a factor 2^t to $I[t]\equiv2^tI$, and if we enlarge the tubes $\widehat{I}$ transversally (meaning perpendicular to $\Phi(c_I)$) by a factor of 2^r to $\widehat{I}[r]$, then we obtain the estimate,*

$$\left|\sum_{I\in\mathcal{G}_s[U]}\left\langle T\mathbf{1}_{I[t]}f,\mathbf{1}_{\widehat{I}[r]}g\right\rangle\right|\leq C2^{t\frac{n-1}{p'}}2^{r\frac{n-1}{p}}2^{-s\varepsilon_{p,n}}\left\|f\right\|_{L^p(\mathbb{R}^{n-1})}\left\|g\right\|_{L^{p'}(\mathbb{R}^n)}\ .$$

Proof. Apply the above argument and use $\left(\left|\widehat{I}[r]\right|\left|I[t]\right|^{p-1}\right)^{\frac{1}{p}}=2^{r\frac{n-1}{p}}2^{t\frac{n-1}{p'}}\left(\left|\widehat{I}\right||I|^{p-1}\right)^{\frac{1}{p}}$. □

Remark 52. *The corollary is easily modifiec to include the smooth Alpert wavelets case,*

$$\sum_{I\in\mathcal{G}_s[U]}\sum_{J\in\mathcal{D}:\ J\subset\widehat{I}}\left|\left\langle T\bigtriangleup_{I;\kappa}^{n-1,\eta}f,\bigtriangleup_{J;\kappa}^{n,\eta}g\right\rangle\right|\leq C'_\eta2^{-s\varepsilon_{p,n}}\left\|f\right\|_{L^p(\mathbb{R}^{n-1})}\left\|g\right\|_{L^{p'}(\mathbb{R}^n)}\ . \tag{9.17}$$

We now turn to obtaining the stronger norm estimate

$$\sum_{s=1}^{\infty}\left\|T\left(\mathsf{Q}_U^s\right)^{\spadesuit}f\right\|_{L^p(A_+(0,2^{2s}))}\lesssim\left\|f\right\|_{L^p(\mathbb{R}^{n-1})}\ , \tag{9.18}$$

and for this, we must consider the smooth Alpert wavelets case (9.17), so that we can use integration by parts in the x-variable in the expanded pipes.

Expanded pipes

Consider an expanded truncated pipe $P_{s,0}^{I_0}[r]$. For $r \gg 0$, we claim that the wavelength on I_0 in the inner product is much smaller than the diameter 2^{-s} of I_0, and so we can use integration by parts to gain a geometric decay factor of $C_N 2^{-rN}$ for all $N \geq 1$. Indeed, for $\xi \in J$ with $J \subset P_{s,0}^{I_0}[r]$ and $0 \leq r \lesssim s$, the wavelength of the exponential factor $e^{-i\Phi(x)\cdot\xi}$ is roughly $\frac{1}{|\xi|} \approx \frac{1}{2^{2s}}$, and referring to (9.13), we see that the tilted depth of I_0 in the direction ξ, is roughly $\ell(I)\sin\theta$, where $\sin\theta = \frac{|\xi'|}{|\xi|} \approx \frac{2^{r+s}}{2^{2s}}$. Altogether then,

$$\text{tilted depth} \approx \ell(I)\sin\theta \approx 2^{-s}\frac{2^{r+s}}{2^{2s}} = 2^r\frac{1}{2^{2s}} = 2^r\,\text{wavelength},$$

and so the exponential factor $e^{-i\Phi(x)\cdot\xi}$ oscillates roughly 2^r times as x traverses I_0.

Thus

$$\left\langle T\,\triangle_{I;\kappa}^{n-1,\eta} f, \triangle_{J;\kappa}^{n,\eta} g\right\rangle = \int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^{n-1}} e^{-i\Phi(x)\cdot\xi}\,\triangle_{I;\kappa}^{n-1,\eta} f(x)\,dx\right\}\triangle_{J;\kappa}^{n,\eta} g(\xi)\,d\xi,$$

where for $\xi \in J$ and $J \subset P_{s,0}^{I_0}(r)$, the integral in braces satisfies,

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} e^{-i\Phi(x)\cdot\xi}\,\triangle_{I;\kappa}^{n-1,\eta} f(x)\,dx &= \int_{\mathbb{R}^{n-1}}\left(\frac{1}{-i\partial_x(\Phi(x)\cdot\xi)}\partial_x\right)^N e^{-i\Phi(x)\cdot\xi}\,\triangle_{I;\kappa}^{n-1,\eta} f(x)\,dx \\ &= (-1)^N\int_{\mathbb{R}^{n-1}} e^{-i\Phi(x)\cdot\xi}\left(\partial_x\frac{1}{-i\Phi'(x)\cdot\xi}\right)^N \triangle_{I;\kappa}^{n-1,\eta} f(x)\,dx,\end{aligned}$$

and hence is dominated in modulus by $C_N 2^{-rN}\int\left|\partial^N\,\triangle_{I;\kappa}^{n-1,\eta} f(x)\right|dx$ since

$$|\Phi'(x)\cdot\xi| \approx |\xi'| \approx 2^{r+s} \qquad \left(\text{also } \approx \frac{1}{\ell(I)}\frac{\text{tilted depth}}{\text{wavelength}} \gtrsim 2^{r+s}\right), \qquad \text{for } \xi \in P_{s,0}^{I_0}(r).$$

In conclusion, for any cube $I \in \mathcal{G}_s[S]$ we have

$$\left|\int_{\mathbb{R}^{n-1}} e^{-i\Phi(x)\cdot\xi}\,\triangle_{I;\kappa}^{n-1,\eta} f(x)\,dx\right| \lesssim C_N 2^{-(r+s)N}\int_{\mathbb{R}^{n-1}}\left|\partial^N\,\triangle_{I;\kappa}^{n-1,\eta} f(x)\right|dx, \qquad \xi \in P_{s,0}^{I}[r]. \tag{9.19}$$

Plugging this estimate back into the inner product gives

$$\begin{aligned}\left|\left\langle T\,\triangle_{I;\kappa}^{n-1,\eta} f, \triangle_{J;\kappa}^{n,\eta} g\right\rangle\right| &\leq \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^{n-1}} e^{-i\Phi(x)\cdot\xi}\,\triangle_{I;\kappa}^{n-1,\eta} f(x)\,dx\right|\left|\triangle_{J;\kappa}^{n,\eta} g(\xi)\right|d\xi \\ &\lesssim C_N 2^{-(r+s)N}\left(\int_{\mathbb{R}^{n-1}}\left|\partial^N\,\triangle_{I;\kappa}^{n-1,\eta} f\right|\right)\left(\int_{\mathbb{R}^n}\left|\triangle_{J;\kappa}^{n,\eta} g\right|\right).\end{aligned} \tag{9.20}$$

For use later on, we note that for any $K \in \mathcal{G}[S]$ with $\ell(K) \geq 2^{-s}$, we can sum over $I \in \mathcal{G}_s[K]$ in (9.19) to obtain

$$\left|\int_{\mathbb{R}^{n-1}} e^{-i\Phi(x)\cdot\xi}\left(\mathsf{Q}_K^s\right)^{\spadesuit} f(x)\,dx\right| \lesssim C_N 2^{-(r+s)N}\int_{\mathbb{R}^{n-1}}\left|\partial^N\left(\mathsf{Q}_K^s\right)^{\spadesuit} f(x)\right|dx, \qquad \xi \in P_{s,0}^{K}[r], \tag{9.21}$$

and with a similar estimate of the corresponding inner product.

We now apply the argument used above for bounding

$$Z_{s,0}^{\mathbf{1}} \equiv \left|\sum_{I\in\mathcal{G}_s[U]}\sum_{J\subset T_{s,0}^I}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^{n-1}} e^{-i\Phi(x)\cdot\xi}\,\triangle_{I;\kappa}^{n-1,\eta} f(x)\,dx\right\}\triangle_{J;\kappa}^{n,\eta} g(\xi)\,d\xi\right|,$$

to the expanded truncated pipes $P^I_{s,0}[r]$ in place of the tubes $T^I_{s,0}$, to obtain from Corollary 51 and the estimate (9.19), that

$$\left\| T \triangle^{n-1,\eta}_{I;\kappa} f \right\|_{L^p\left(P^I_{s,0}[r]\right)} = \left(\int_{P^I_{s,0}[r]} \left| \int_{\mathbb{R}^{n-1}} e^{-i\Phi(x)\cdot\xi} \triangle^{n-1,\eta}_{I;\kappa} f(x)\, dx \right|^p d\xi \right)^{\frac{1}{p}} \tag{9.22}$$
$$\leq \left| P^I_{s,0}[r] \right|^{\frac{1}{p}} |I|^{\frac{1}{p'}} \left(C_N 2^{-(r+s)Np} \int_{\mathbb{R}^{n-1}} \left| \partial^N \triangle^{n-1,\eta}_{I;\kappa} f(x) \right|^p dx \right)^{\frac{1}{p}}$$
$$\leq C_N 2^{-(r+s)N} 2^{r\frac{n-1}{p}} \left| P^I_{s,0} \right|^{\frac{1}{p}} |I|^{\frac{1}{p'}} \left(\int_{\mathbb{R}^{n-1}} \left| \partial^N \triangle^{n-1,\eta}_{I;\kappa} f(x) \right|^p dx \right)^{\frac{1}{p}}$$
$$\leq C_N 2^{-r\left(N-\frac{n-1}{p}\right)} 2^{-s\varepsilon_{p,n}} 2^{-sN} \left\| \partial^N \triangle^{n-1,\eta}_{I;\kappa} f \right\|_{L^p(\mathbb{R}^{n-1})} .$$

Thus

$$\left(\sum_{I\in\mathcal{G}_s[U]} \left\| T \triangle^{n-1,\eta}_{I;\kappa} f \right\|^p_{L^p\left(P^I_{s,0}[r]\right)} \right)^{\frac{1}{p}} \lesssim C_N 2^{-r\left(N-\frac{n-1}{p}\right)} 2^{-s\varepsilon_{p,n}} \left(\sum_{I\in\mathcal{G}_s[U]} 2^{-sNp} \left\| \partial^N \triangle^{n-1,\eta}_{I;\kappa} f \right\|^p_{L^p(\mathbb{R}^{n-1})} \right)^{\frac{1}{p}}$$
$$\lesssim C_N 2^{-r\left(N-\frac{n-1}{p}\right)} 2^{-s\varepsilon_{p,n}} \|f\|^p_{L^p(\mathbb{R}^{n-1})} ,$$

and so also,

$$Z^{\mathbf{1}}_{s,0}[r] \equiv \left| \sum_{I\in\mathcal{G}_s[U]} \sum_{J\subset P^I_s[r]} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^{n-1}} e^{-i\Phi(x)\cdot\xi} \triangle^{n-1,\eta}_{I;\kappa} f(x)\, dx \right\} \triangle^{n,\eta}_{J;\kappa} g(\xi)\, d\xi \right| \tag{9.23}$$
$$\leq \sum_{I\in\mathcal{G}_s[S]} \left\| T \triangle^{n-1,\eta}_{I;\kappa} f \right\|_{L^p\left(P^I_{s,0}[r]\right)} \|g\|_{L^{p'}\left(P^I_{s,0}[r]\right)}$$
$$\leq \left(\sum_{I\in\mathcal{G}_s[U]} \left\| T \triangle^{n-1,\eta}_{I;\kappa} f \right\|^p_{L^p\left(\widehat{I}\right)} \right)^{\frac{1}{p}} \left(\sum_{I\in\mathcal{G}_s[U]} \|g\|^{p'}_{L^{p'}\left(P^I_{s,0}[r] I\right)} \right)^{\frac{1}{p'}}$$
$$\leq C_N 2^{-r\left(N-\frac{n-1}{p}\right)} 2^{-s\varepsilon_{p,n}} \|f\|_{L^p(\mathbb{R}^{n-1})} \|g\|_{L^{p'}(\mathbb{R}^n)} .$$

Norm estimate

We can extend this inner product estimate to a norm estimate by duality. Choose an appropriate function g_s with $\|g_s\|_{L^{p'}(\mathbb{R}^n)} = 1$ and

$$\left\langle T\left(\mathsf{Q}^s_U\right)^{\spadesuit} f, g_s \right\rangle = \left\| T\left(\mathsf{Q}^s_U\right)^{\spadesuit} f \right\|_{L^p\left(\bigcup_{I\in\mathcal{G}_s[U]} \left\{ T^I_s \cup \bigcup_{r\geq 0} P^I_s[r] \right\} \right)} ,$$

and then with $N > \frac{n-1}{p}$ and $p > \frac{2n}{n-1}$, sum in r and s to obtain

$$\sum_{s=1}^{\infty} \left\| T\left(\mathsf{Q}^s_U\right)^{\spadesuit} f \right\|_{L^p(A_+(0,2^{2s}))} \leq \sum_{s=1}^{\infty} \left\| T\left(\mathsf{Q}^s_U\right)^{\spadesuit} f \right\|_{L^p\left(\bigcup_{I\in\mathcal{G}_s[U]} \left\{ T^I_s \cup \bigcup_{r\geq 0} P^I_s[r] \right\} \right)}$$
$$= \sum_{s=1}^{\infty} \left| \left\langle T\left(\mathsf{Q}^s_U\right)^{\spadesuit} f, g_s \right\rangle \right| \leq \sum_{s=1}^{\infty} \sum_{r=0}^{\infty} C_N 2^{-r\left(N-\frac{n-1}{p}\right)} 2^{-s\varepsilon_{p,n}} \|f\|_{L^p(\mathbb{R}^{n-1})} \|g_s\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^{n-1})} ,$$

which is (9.18).

9.3.2. *The case* $w=s$. In this case we need to take expectation. Since each fixed cube J in the upper quarter annulus $A_+(0,2^s)$ belongs to the truncated pipe $P^I_{s,s}\equiv P^I_s\cap L^I_s$ for essentially all $I\in\mathcal{G}_s[S]$, we get

$$\begin{aligned} Z^{\mathbf{a}}_{s,s} &= \left|\sum_{I\in\mathcal{G}_s[U]}\sum_{J\subset P^I_{s,s}}\int_{\mathbb{R}^n}\left\{\int e^{-i\Phi(x)\cdot\xi}\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}^s_U\right)^{\spadesuit}f(x)\,dx\right\}\triangle^{n,\eta}_{J;\kappa}g(\xi)\,d\xi\right| \\ &\approx \left|\sum_{Q_0}\left\langle T\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}^s_{Q_0}\right)^{\spadesuit}f,\mathsf{P}^{n,\eta}_{Q_0^*,s;\kappa}g\right\rangle\right|\lesssim\sum_{Q_0}\left\|T\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}^s_{Q_0}\right)^{\spadesuit}f\right\|_{L^p}\left\|\mathsf{P}^{n,\eta}_{Q_0^*,s;\kappa}g\right\|_{L^{p'}}, \end{aligned}$$

where $\mathsf{Q}^s_{Q_0}=\sum_{I\in\mathcal{G}_s[Q_0]}\triangle^{n-1}_{I;\kappa}$ and $\mathsf{P}^{n,\eta}_{Q_0^*,s;\kappa}=\sum_{J\in\mathcal{D}_k[Q_0^*]}\triangle^{n-1,\eta}_{I;\kappa}$, and where Q_0 ranges over a bounded number of cubes in S with side length approximately 1. Also note that

$$\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}^s_{Q_0}\right)^{\spadesuit}f=S_{\kappa,\eta}\mathcal{A}_{\mathbf{a}}\sum_{I\in\mathcal{G}_s[Q_0]}\left\langle\left(S_{\kappa,\eta}\right)^{-1}f,h^{n-1}_{I;\kappa}\right\rangle h^{n-1}_{I;\kappa}=\sum_{I\in\mathcal{G}_s[Q_0]}a_I\,\triangle^{n-1,\eta}_{I;\kappa}\ .$$

Now we apply just part of the estimate (9.12), which followed from Proposition 32, to obtain

$$\mathbb{E}^{\mu}_{2^{\mathcal{G}_s[U]}}Z^{\mathbf{a}}_{s,s}\lesssim\left(\mathbb{E}^{\mu}_{2^{\mathcal{G}_s[U]}}\left\|T_S\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}^s_{Q_0}\right)^{\spadesuit}f\right\|^p_{L^p(B(0,2^s))}\right)^{\frac{1}{p}}\left\|\mathsf{P}^{n,\eta}_{Q_0^*,s;\kappa}g\right\|_{L^{p'}}\lesssim 2^{-\varepsilon_{p,n}s}\|f\|_{L^p}\|g\|_{L^{p'}}\ ,$$

for $p>\frac{2n}{n-1}$ and $m=s\in\mathbb{N}$.

We do not need to make use of expanded pipes in this case, due to the small size of the ball $B(0,2^s)$.

9.4. **The general case** $0\le w\le s$ **via square functions.** In this subsection we prove the average norm estimate for each $s\in\mathbb{N}$,

$$\mathbb{E}^{\mu}_{2^{\mathcal{G}[U]}}\left\|T\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}^s_U\right)^{\spadesuit}f\right\|_{L^p(B(0,2^{2s}))}\lesssim 2^{-\varepsilon_{n,p}s}\|f\|^p_{L^p}\ ,\qquad\text{for }p>\frac{2n}{n-1}. \tag{9.24}$$

It will be convenient to pass back and forth between average norm estimates and square function estimates using Khintchine's inequalities. For example (9.24) is equivalent to,

$$\left\|\mathcal{S}^{\eta}_{T,s}f\right\|_{L^p(B(0,2^{2s}))}\lesssim 2^{-\varepsilon_{n,p}s}\|f\|_{L^p}\ ,\qquad\text{for }p>\frac{2n}{n-1}, \tag{9.25}$$

where

$$\mathcal{S}^{\eta}_{T,s}f\equiv\left(\sum_{I\in\mathcal{G}_s[U]}\left|T\,\triangle^{n-1,\eta}_{I;\kappa}f\right|^2\right)^{\frac{1}{2}} \tag{9.26}$$

is the square function associated with the random decomposition

$$T\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}^s_U\right)^{\spadesuit}f=\sum_{I\in\mathcal{G}_s[U]}a_IT\,\triangle^{n-1,\eta}_{I;\kappa}f,\qquad\text{for each }\mathbf{a}\in 2^{\mathcal{G}[U]}.$$

Finally, for the purpose of establishing (9.25), it suffices to prove the following inequality for each $0\le w\le s$,

$$\left\|\mathcal{S}^{\eta}_{T,s}f\right\|_{L^p(A_+(0,2^{2s-w}))}\lesssim 2^{-s\varepsilon_{p,n}}\|f\|_{L^p}\ , \tag{9.27}$$

with implied constant independent of s and w. Indeed, the case $w=s$, which is $\left\|\mathcal{S}^{\eta}_{T,s}f\right\|^p_{L^p(B_+(0,2^s))}\lesssim 2^{-s\varepsilon_{p,n}}\|f\|^p_{L^p}$, follows from applying Khinchine's inequality to the model inequality (5.3), and now we finish the proof using the decomposition

$$B_+\left(0,2^{2s}\right)=B_+\left(0,2^s\right)\cup\bigcup_{w=0}^{s-1}A_+\left(0,2^{2s-w}\right).$$

We will prove (9.27) in four steps, the first two being local estimates requiring *probabilistic* arguments, and the second two being global estimates for expanded pipes that require *deterministic* arguments. The probabilistic local estimates are used to control the sums over cubes $I\in\mathcal{G}_s[K]$ which are close together, while the global deterministic estimates are used to control the sums of cubes $K \in \mathcal{G}_{s-w}[S]$ which are farther apart.

9.4.1. *Step 1: local probabilistic argument.* Here we prove the local square function inequality,

$$\begin{aligned} \left\| \mathcal{S}^{\eta}_{T,s} \left(\mathsf{Q}^s_K \right)^{\spadesuit} f \right\|^p_{L^p(A_+(0,2^{2s-w}))} &\lesssim 2^{-s\varepsilon_{p,n}} \left\| \left(\mathsf{Q}^s_K \right)^{\spadesuit} f \right\|^p_{L^p(\mathbb{R}^{n-1})}, \quad \text{for all } K \in \mathcal{G}_{s-w}[U], \\ \sum_{s=1}^{\infty} \left\| \mathcal{S}^{\eta}_{T,s} \left(\mathsf{Q}^s_K \right)^{\spadesuit} f \right\|^p_{L^p(A_+(0,2^{2s-w}))} &\lesssim \left\| \left(\mathsf{Q}^s_K \right)^{\spadesuit} f \right\|^p_{L^p(\mathbb{R}^{n-1})}, \quad \text{for all } K \in \mathcal{G}_{s-w}[U], \end{aligned}$$

which by Khintchine's inequalities is equivalent to the local average expectation inequality,

$$\mathbb{E}^{\mu}_{2^{\mathcal{G}[U]}} \sum_{s=1}^{\infty} \left\| T \left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}^s_K \right)^{\spadesuit} f \right\|^p_{L^p(B(0,2^{2s-w}))} \lesssim \left\| \left(\mathsf{Q}^s_K \right)^{\spadesuit} f \right\|^p_{L^p(\mathbb{R}^{n-1})}, \quad \text{for all } K \in \mathcal{G}_{s-w}[U].$$

Consider $(I,J) \in \mathcal{R}^{k,w}_s$, i.e. $I \in \mathcal{G}_s[S]$, $\ell(J) = 2^k$ and $J \subset P^I_{s,w}$. Recall that $T^I_{s,w}$ is the tube given by the convex hull of the pipe $P^I_{s,w}$. For $0 < w < s$, these tubes have bounded overlap approximately $2^{w(n-1)}$. Thus for each $K \in \mathcal{G}_{s-w}$ we can define a tube $T^{K,\natural}_{s,w} \equiv \bigcup_{I \in \mathcal{G}_s[K]} T^I_{s,w}$ consisting of all the tubes $T^I_{s,w}$ with $I \subset K$. Note that each tube $T^I_{s,w}$ has dimensions $C_1 2^s \times 2^{2s-w}$, and due to the $2^{w(n-1)}$ overlap, each of the tubes $T^{K,\natural}_{s,w}$ also has dimensions $C_2 2^s \times 2^{2s-w}$, but with a larger constant C_2. Finally, note that the union $\bigcup_{K \in \mathcal{G}_{s-w}} T^{K,\natural}_{s,w}$ of these tubes covers the upper quarter annulus $A_+(0, 2s-w)$ with *bounded* overlap. Indeed, the tubes $T^{K,\natural}_{s,w}$ are comparable to any of the tubes $T^I_{s,w}$ with $I \subset K$, and it is this last property that motivated grouping the $I's$ into cubes K and defining $T^{K,\natural}_{s,w}$ as we did above.

We begin with the following more elementary local average inequality for $0 \le w \le s$, in which we restrict the integration over $\mathbb{R}^n$ to the tubes $T^{K,\natural}_{s,w}$,

$$\mathbb{E}^{\mu}_{2^{\mathcal{G}_s[S]}} \left\| T_S \left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}^s_K \right)^{\spadesuit} f \right\|^p_{L^p\left(T^{K,\natural}_{s,w}\right)} \lesssim 2^{-(2s-w)p\varepsilon_{p,n}} \left\| f \right\|^p_{L^p(\mathbb{R}^{n-1})}, \quad \text{for } p > \frac{2n}{n-1}. \tag{9.28}$$

To prove this, we consider the L^2 and average L^4 bounds separately and then interpolate.

Step 1(a): local L^2 estimate

We first compute the norm of $\Lambda^{2s}_{\mathsf{Q}^s_K}$ from $L^2(\lambda_{n-1})$ to $L^2\left(T^{K,\natural}_{s,w}\right)$, where we recall that

$$\Lambda^{2s}_{\mathsf{Q}^s_K} f \equiv \left(\widehat{\left(\mathsf{Q}^s_K \right)^{\spadesuit} f} \right)_{\Phi,2s}.$$

Consistent with (5.4), we write and

$$\begin{aligned} f^s_K &\equiv \left(\mathsf{Q}^s_K \right)^{\spadesuit} f, \\ \left(f^s_K \right)_{\Phi} &\equiv \Phi_* \left[\left(\mathsf{Q}^s_K \right)^{\spadesuit} f \right] = \sum_{I \in \mathcal{G}_s(K)} \Phi_* \left[\left(\triangle^{n-1}_{I;\kappa} \right)^{\spadesuit} f \right] = \sum_{I \in \mathcal{G}_s(K)} f^I_{\Phi}, \\ \left(f^s_K \right)_{\Phi,r} &= \sum_{I \in \mathcal{G}_s(K)} f^I_{\Phi,r}. \end{aligned} \tag{9.29}$$

For $I_0 \in \mathcal{G}_s[K]$, whose normal is $\mathbf{e}_n$, we will use the rectangular convolver $\varphi_{s,2s-w}(z)$ that has dimensions $2^{-s} \times 2^{w-2s}$, and we will multiply by a modulation $m(z)$ that translates the tube $[-2^s, 2^s]^{n-1} \times [-2^{2s-w}, 2^{2s-w}]$ to be positioned near $T^{K,\natural}_{s,w}$. For convenience we momentarily set

$$\psi(z) \equiv m(z) \varphi_{s,2s-w}(z).$$

We then have with $f_K^s = (\mathsf{Q}_K^s)^{\spadesuit} f$,

$$\left\|\Lambda_{\mathsf{Q}_K^s}^{2s} f\right\|_{L^2\left(\left|\widehat{\psi}\right|^2\lambda_n\right)}^2 = \int_{\mathbb{R}^n} \left|\widehat{(f_K^s)_{\Phi,2s}}(\xi)\right|^2 \left|\widehat{\psi}(\xi)\right|^2 d\xi = \int_{\mathbb{R}^n} \overline{\widehat{(f_K^s)_{\Phi,2s}} * \psi(\xi)}\ \widehat{(f_K^s)_{\Phi,2s}} * \psi(\xi)\, d\xi$$

$$= \sum_{I,J\in\mathcal{G}_s[K]} \int_{\mathbb{R}^n} \overline{\widehat{f_{\Phi,2s}^I *} \psi(\xi)}\ \widehat{f_{\Phi,2s}^J *} \psi(\xi)\, d\xi = \sum_{I,J\in\mathcal{G}_s[K]} \int_S \overline{f_{\Phi,2s}^I * \psi(x)}\ \left(f_{\Phi,2s}^J * \psi\right)(x)\, dx.$$

Note first that the supports of $f_{\Phi,2s}^I * \psi$ and $f_{\Phi,2s}^J * \psi$ are essentially disjoint unless $I \sim J$. Next, if we define

$$I_0^* \equiv \left(\left[-2^{-s}, 2^{-s}\right]^{n-1} \times \left[-2^{w-2s}, 2^{w-2s}\right]\right) + \mathbf{e}_n\ ,$$

and I^* by rotation, then we have

$$\left|f_{\Phi,2s}^I * \psi(z)\right| \lesssim \left|\left\langle S_{\kappa,\eta}^{-1} f, h_{I;\kappa}^{n-1}\right\rangle\right| 2^{2s-w} 2^{s\frac{n-1}{2}} \mathbf{1}_{I^*}(z)\,,$$

since

$$\left|f_{\Phi,2s}^I * \psi\right| \approx \left|f_\Phi^I * \psi\right| \lesssim \left\|\frac{df_\Phi^I}{d\sigma_{n-1}}\right\|_\infty \left(\mathbf{1}_{\Phi(I)}\sigma_{n-1}\right) * \varphi_{s,2s-w}(z) \approx \left|\left\langle S_{\kappa,\eta}^{-1} f, h_{I;\kappa}^{n-1}\right\rangle\right| 2^{s\frac{n-1}{2}} (\text{density})\, \mathbf{1}_{I^*}(z)\,,$$

where the quantity density satisfies,

$$\begin{aligned} (\text{density})\, 2^{-s(n-1)} 2^{w-2s} &= (\text{density})\, |I^*| = \left\|\mathbf{1}_{\Phi(I)}\sigma_{n-1}\right\| = 2^{-s(n-1)} \\ &\Longrightarrow \text{density} = \frac{2^{-s(n-1)}}{2^{-s(n-1)}2^{w-2s}} = 2^{2s-w}. \end{aligned}$$

Altogether then, using $|I^*| = 2^{-s(n-1)}2^{w-2s}$, we have using (9.29) that

$$\left\|\Lambda_{\mathsf{Q}_K^s}^{2s} f\right\|_{L^2\left(\left|\widehat{\psi}\right|^2\lambda_n\right)}^2 \lesssim \int_{\mathbb{R}^n} \left|(f_K^s)_{\Phi,2s} * \psi(\xi)\right|^2 d\xi = \sum_{I\in\mathcal{G}_s[K]} \int_{\mathbb{R}^n} \left|f_{\Phi,2s}^I * \psi(\xi)\right|^2 d\xi \tag{9.30}$$

$$\lesssim \sum_{I\in\mathcal{G}_s[K]} \int_{\mathbb{R}^n} \left|\left|\left\langle S_{\kappa,\eta}^{-1} f, h_{I;\kappa}^{n-1}\right\rangle\right| 2^{2s-w} 2^{s\frac{n-1}{2}} \mathbf{1}_{I^*}(\xi)\right|^2 d\xi \lesssim \sum_{I\in\mathcal{G}_s[K]} \left|\left\langle S_{\kappa,\eta}^{-1} f, h_{I;\kappa}^{n-1}\right\rangle\right|^2 \left(2^{2s-w} 2^{s\frac{n-1}{2}}\right)^2 |I^*|$$

$$= 2^{4s-2w} 2^{s(n-1)} 2^{-s(n-1)} 2^{w-2s} \sum_{I\in\mathcal{G}_s[K]} \left|\left\langle S_{\kappa,\eta}^{-1} f, h_{I;\kappa}^{n-1}\right\rangle\right|^2 = 2^{2s-w} \sum_{I\in\mathcal{G}_s[K]} \left|\left\langle S_{\kappa,\eta}^{-1} f, h_{I;\kappa}^{n-1}\right\rangle\right|^2 \lesssim 2^{2s-w} \left\|f_K^s\right\|_{L^2(S)}^2 .$$

Step 1(b): local average L^4 estimate

We run the argument in Subsection 5.2 up until the estimate for $\Omega_t = \Omega_t[K]$, where $2^{-t} \approx \operatorname{dist}(I,J)$ for $I,J \in \mathcal{G}_s[K]$, i.e. $2^{-t} \lesssim \ell(K) = 2^{w-s}$ or $s-w \leq t \leq s$. It is this restriction to large t that yields the geometric gain needed for the average L^4 estimate when $I,J \in \mathcal{G}_s[K]$. Then for $s-w < t < s$, and *with notation as in Subsection 5.2*, we have

$$\begin{aligned} \Omega_t[K] &\lesssim \sum_{I,J\in\mathcal{G}_s[K]:\ \operatorname{dist}(I,J)\approx 2^{-t}} 2^{-s(n-2)} 2^t \left|\left\langle (S_{\kappa,\eta})^{-1} f, h_{I;\kappa}\right\rangle \left\langle (S_{\kappa,\eta})^{-1} f, h_{J;\kappa}\right\rangle\right|^2 \\ &\lesssim 2^{-s(n-2)} 2^t \sum_{I,J\in\mathcal{G}_s[K]:\ \operatorname{dist}(I,J)\approx 2^{-t}} \left|\left\langle (S_{\kappa,\eta})^{-1} f, h_{I;\kappa}\right\rangle\right|^4 \\ &\lesssim 2^{-s(n-2)} 2^t 2^{(s-t)(n-1)} \sum_{I\in\mathcal{G}_s[K]} \left|\left\langle (S_{\kappa,\eta})^{-1} f, h_{I;\kappa}\right\rangle\right|^4 = 2^{-t(n-2)} 2^{-s(n-2)} \left\|\mathsf{Q}_K^s (S_{\kappa,\eta})^{-1} f\right\|_{L^4(S)}^4 , \end{aligned}$$

which gives

$$\begin{aligned} \sum_{t=s-w}^s \Psi_t[K] &\lesssim \sum_{t=s-w}^s \Omega_t[K] \lesssim \sum_{t=s-w}^s 2^{-t(n-2)} 2^{-s(n-2)} \left\|\mathsf{Q}_K^s (S_{\kappa,\eta})^{-1} f\right\|_{L^4(S)}^4 \\ &\approx 2^{-(s-w)(n-2)} 2^{-s(n-2)} \left\|\mathsf{Q}_K^s (S_{\kappa,\eta})^{-1} f\right\|_{L^4(S)}^4 = 2^{-(2s-w)(n-2)} \left\|\mathsf{Q}_K^s (S_{\kappa,\eta})^{-1} f\right\|_{L^4(S)}^4 . \end{aligned}$$

Similarly we obtain

$$\Psi \lesssim 2^{-(2s-w)(n-2)} \left\| \mathsf{Q}_K^s \left(S_{\kappa,\eta}\right)^{-1} f \right\|_{L^4(S)}^4 ,$$

and adding these last two inequalities gives, $\left\| \Lambda_{\mathsf{Q}_K^s}^{2s} f \right\|_{L^2\left(\left|\widehat{\psi}\right|^2 \lambda_n\right)}^2$

$$\mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\| \Lambda_{\mathcal{A}_{\mathbf{a}} \mathsf{Q}_K^s}^{2s} f \right\|_{L^4\left(\left|\widehat{\psi}\right|^2 \lambda_n\right)}^4 \lesssim 2^{-(2s-w)(n-2)} \|f\|_{L^4(S)}^4 . \tag{9.31}$$

Step 1(c): local interpolation

Collecting the bounds (9.30) and (9.31) gives,

$$\begin{aligned} \left\| \Lambda_{\mathcal{A}_{\mathbf{a}} \mathsf{Q}_K^s}^{2s} f \right\|_{L^2\left(\left|\widehat{\psi}\right|^2 \lambda_n\right)}^2 &\lesssim 2^{\frac{2s-w}{2}} \|f\|_{L^2(K)}^2 , \\ \mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\| \Lambda_{\mathcal{A}_{\mathbf{a}} \mathsf{Q}_K^s}^{2s} f \right\|_{L^4\left(\left|\widehat{\psi}\right|^2 \lambda_n\right)}^4 &\lesssim 2^{-\frac{2s-w}{2}\frac{n-2}{2}} \|f\|_{L^4(S)}^4 . \end{aligned}$$

Now we claim that an application of the interpolation Lemma 34 yields,

$$\mathbb{E}_{2^{\mathcal{G}}}^{\mu} \left\| \Lambda_{\mathcal{A}_{\mathbf{a}} \mathsf{Q}_K^s}^{2s} f \right\|_{L^p\left(\left|\widehat{\psi}\right|^2 \lambda_n\right)} \lesssim 2^{-(2s-w)\varepsilon'_{p,n}} \|f\|_{L^p(\mathbb{R}^{n-1})} , \qquad \text{for } p > \frac{2n}{n-1}. \tag{9.32}$$

Indeed, the calculation at the end of the proof of Lemma 34 shows that if $p > \frac{2n}{n-1}$, then (with notation as in that proof) $\theta = \frac{4}{p} - 1$ and so

$$\left[2^{-\frac{2s-w}{2}\frac{n-2}{2}}\right]^{1-\theta} \left[2^{\frac{2s-w}{2}}\right]^{\theta} = 2^{-\frac{2s-w}{2}\frac{n-2}{2}} 2^{\left(\frac{2s-w}{2} + \frac{2s-w}{2}\frac{n-2}{2}\right)\theta} = 2^{-\frac{2s-w}{2}\frac{n-2}{2}} 2^{\left(\frac{2s-w}{2}\frac{n}{2}\right)\theta} = 2^{-(2s-w)\varepsilon'_{p,n}},$$

where

$$\begin{aligned} \varepsilon'_{p,n} &\equiv \frac{1}{2s-w} \left\{ \frac{2s-w}{2}\frac{n-2}{2} - \left(\frac{2s-w}{2}\frac{n}{2}\right)\left(\frac{4}{p} - 1\right) \right\} \\ &= \frac{n-2}{4} - \frac{n}{4}\left(\frac{4}{p} - 1\right) = \frac{n-1}{2} - \frac{n}{p} = \frac{n-1}{2p}\left(p - \frac{2n}{n-1}\right). \end{aligned}$$

This completes our proof of (9.28) in *Step 1*.

9.4.2. *Step 2: local expanded probabilistic argument.* Now we turn to proving the expanded analogue of (9.28) given by,

$$\mathbb{E}_{2^{\mathcal{G}_s[U]}}^{\mu} \left\| T\left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_K^s\right)^{\spadesuit} f \right\|_{L^p\left(P_{s,w}^K[r]\right)}^p \lesssim 2^{-rp\left(N - \frac{n-1}{p'}\right)} 2^{-(2s-w)p\varepsilon_{p,n}} \|f\|_{L^p(\mathbb{R}^{n-1})}^p \tag{9.33}$$

$$\text{for all } K \in \mathcal{G}_{s-w}[S] \text{ and } p > \frac{2n}{n-1},$$

where $\delta > 0$ and $P_{s,w}^K[r]$ is the expanded pipe corresponding to the tube $T_{s,w}^K$. This is proved in the same way as the case of the tube $T_{s,w}^{K,\natural}$ in the previous subsubsection, except that we use the geometric decay in r derived from integration by parts, to compensate the geometric growth in r that arises from the expanded pipes.

We first define $V_{s,w}^K$ to be the vertical cone that is the complement of the union over $0 \leq r \leq s$ of the expanded tubes $T_{s,w}^{K,\natural}(r)$ in the quarter annulus $A_+\left(0, 2^{2s-w}\right)$, and set $V_s \equiv \bigcup_{w=0}^{s} \bigcup_{K \in \mathcal{G}_{s-w}[U]} V_{s,w}^K$. Note that the cone V_s will be 'thin' if the positive constant C_n in Definition 48 is large. Now we repeat the above proof of (9.28), but with expanded pipes $P_{s,w}^K[r]$ in place of the tube $T_{s,w}^K$, to get (9.33). Indeed, the L^2 and average L^4 estimates (9.30) and (9.31) are now multiplied by an additional factor $C_\delta 2^{-r\delta}$ for some $\delta > 0$, which percolates through the interpolation to give (9.33).

However, we must choose the constant C_n in Definition 48 to be possibly even larger than it already is. Namely, given a small positive constant ε satisfying $0 < \varepsilon < \varepsilon_{p,n}$, choose C_n such that the vertical cone V_s

is so thin that the **Direct Argument** in Subsubsection 9.3.1 produces a bound that is $C2^{\varepsilon s}$ times as large as that obtained in Subsubsection 9.3.1,

$$\sup_{\mathbf{a}\in 2^{\mathcal{G}_s[U]}} \left|\left\langle T\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}_K^s\right)^{\spadesuit} f, \mathsf{P}_{V_s}^{n,\eta} g\right\rangle\right| \lesssim 2^{\varepsilon s} 2^{-\varepsilon'_{p,n} s} \|f\|_{L^p} \|g\|_{L^{p'}} . \tag{9.34}$$

This bound will prove to be an acceptable estimate if we choose $\varepsilon'_{p,n} > \varepsilon > 0$.

Next we adapt the arguments surrounding (9.32) and (9.22) to conclude that

$$\mathbb{E}^{\mu}_{2^{\mathcal{G}[U]}} \left\|T\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}_K^s\right)^{\spadesuit} f\right\|_{L^p\left(P_{s,w}^K[r]\right)} \lesssim 2^{-\varepsilon_{p,n}s} \left\|\partial^N \left(\mathsf{Q}_K^s\right)^{\spadesuit} f\right\|_{L^p[\mathbb{R}^{n-1}]} , \qquad \text{for } K\in\mathcal{G}_{s-w}[K] \text{ and } p>\frac{2n}{n-1}.$$

Indeed, the following three steps are almost verbatim analogues of Steps 1(a), (b) and (c) above, and we include the details only because of the importance of the estimates. We begin by noting that the analogue of (9.21) in the case $0\le w\le s$ is,

$$\left|\int_{\mathbb{R}^{n-1}} e^{-i\Phi(x)\cdot\xi} \left(\mathsf{Q}_K^s\right)^{\spadesuit} f(x)\,dx\right| \lesssim C_N 2^{-(r+s)N} \int_{\mathbb{R}^{n-1}} \left|\partial^N \left(\mathsf{Q}_K^s\right)^{\spadesuit} f(x)\right| dx, \qquad \text{for } \xi\in P_{s,w}^K[r]. \tag{9.35}$$

Step 2(a): local expanded L^2 estimate

We compute the norm of $\Lambda^{2s}_{\mathsf{Q}_K^s}$ from $L^2\left(\mathbb{R}^{n-1}\right)$ to $L^2\left(P_{s,w}^K[r]\right)$. For $I_0\in\mathcal{G}_s[K]$, whose normal is $\mathbf{e}_n$, we now use the *cylindrical* convolver $\varphi^r_{s,2s-w}(z)$ that has outer dimensions $2^{-s-r}\times 2^{w-2s}$, and we will multiply by a modulation $m(z)$ that translates the pipe whose convex hull is the tube $\left[-2^{s+r},2^{s+r}\right]^{n-1}\times\left[-2^{2s-w},2^{2s-w}\right]$ to be positioned near $P_{s,w}^K[r]$. For convenience we momentarily set

$$\psi(z)\equiv m(z)\,\varphi^r_{s,2s-w}(z).$$

We then have with using (9.29) that,

$$\begin{aligned}\left\|\Lambda^{2s}_{\mathsf{Q}_K^s} f\right\|^2_{L^2\left(\left|\widehat{\psi}\right|^2\lambda_n\right)} &= \int_{\mathbb{R}^n}\left|\widehat{(f_K^s)_{\Phi,2s}}(\xi)\right|^2\left|\widehat{\psi}(\xi)\right|^2 d\xi = \int_{\mathbb{R}^n} \overline{\widehat{(f_K^s)_{\Phi,2s}*\psi}(\xi)}\;\widehat{(f_K^s)_{\Phi,2s}*\psi}(\xi)\,d\xi \\ = \sum_{I,J\in\mathcal{G}_s[K]} &\int_{\mathbb{R}^n} \overline{\widehat{f^I_{\Phi,2s}*\psi}(\xi)}\;\widehat{f^J_{\Phi,2s}*\psi}(\xi)\,d\xi = \sum_{I,J\in\mathcal{G}_s[K]}\int_S \overline{f^I_{\Phi,2s}*\psi(x)}\;\left(f^J_{\Phi,2s}*\psi\right)(x)\,dx.\end{aligned}$$

The supports of $f^I_{\Phi,2s}*\psi$ and $f^J_{\Phi,2s}*\psi$ are essentially disjoint unless $I\sim J$. Next, if we define

$$I_0^*[r]\equiv\left(\left[-2^{-s-r},2^{-s-r}\right]^{n-1}\times\left[-2^{w-2s},2^{w-2s}\right]\right)+\mathbf{e}_n ,$$

and $I^*[r]$ by rotation, then we have

$$\left|f^I_{\Phi,2s}*\psi(z)\right|\lesssim 2^{-(s+r)N}\left|\left\langle S^{-1}_{\kappa,\eta}F^{[N]},h^{n-1}_{I;\kappa}\right\rangle\right| 2^{2s-w}2^{s\frac{n-1}{2}}\mathbf{1}_{I^*[r]}(z),$$

since

$$\begin{aligned}\left|f^I_{\Phi,2s}*\psi\right| &\approx \left|f^I_{\Phi}*\psi\right|\lesssim\left\|\frac{df^I_\Phi}{d\sigma_{n-1}}\right\|_\infty\left(\mathbf{1}_{\Phi(I)}\sigma_{n-1}\right)*\varphi^r_{s,2s-w}(z)\\ &\approx 2^{-(s+r)N}\left|\left\langle S^{-1}_{\kappa,\eta}f,h^{n-1}_{I;\kappa}\right\rangle 2^{sN}\right| 2^{s\frac{n-1}{2}}\,(\text{density})\,\mathbf{1}_{I^*[r]}(z),\end{aligned}$$

upon applying (9.35) to

$$\partial^N\triangle^{n-1,\eta}_{I;\kappa}f=\left\langle S^{-1}_{\kappa,\eta}f,h^{n-1}_{I;\kappa}\right\rangle\partial^N h^{n-1,\eta}_{I;\kappa}.$$

Here the quantity density satisfies,

$$\begin{aligned}(\text{density})\,2^{-(s+r)(n-1)}2^{w-2s} &= (\text{density})\left|I^*[r]\right|=\left\|\mathbf{1}_{\Phi(I^*[r])}\sigma_{n-1}\right\|=2^{-(s+r)(n-1)}\\ &\Longrightarrow \text{density}=\frac{2^{-(s+r)(n-1)}}{2^{-(s+r)(n-1)}2^{w-2s}}=2^{2s-w}.\end{aligned}$$

Altogether then, using $|I^*[r]| = 2^{-(s+r)(n-1)}2^{w-2s}$, we have

$$\left\|\Lambda^{2s}_{\mathsf{Q}^s_K} f\right\|^2_{L^2\left(\left|\widehat{\psi}\right|^2\lambda_n\right)} \lesssim \sum_{I\in\mathcal{G}_s[K]} \int_{\mathbb{R}^n} \left|f^I_{\Phi,2s} * \psi(\xi)\right|^2 d\xi$$

$$\lesssim 2^{-2rN} \sum_{I\in\mathcal{G}_s[K]} \int_{\mathbb{R}^n} \left|\left|\left\langle S^{-1}_{\kappa,\eta} f, h^{n-1}_{I;\kappa}\right\rangle\right| 2^{2s-w} 2^{(s+r)\frac{n-1}{2}} \mathbf{1}_{I^*[r]}(\xi)\right|^2 d\xi$$

$$\lesssim 2^{-2rN} \sum_{I\in\mathcal{G}_s[K]} \left|\left\langle S^{-1}_{\kappa,\eta} f, h^{n-1}_{I;\kappa}\right\rangle\right|^2 \left(2^{2s-w}2^{(s+r)\frac{n-1}{2}}\right)^2 |I^*|$$

$$= 2^{-2rN}2^{4s-2w}2^{(s+r)(n-1)}2^{-s(n-1)}2^{w-2s} \sum_{I\in\mathcal{G}_s[K]} \left|\left\langle S^{-1}_{\kappa,\eta} f, h^{n-1}_{I;\kappa}\right\rangle\right|^2$$

$$= 2^{-2rN}2^{2s-w} \sum_{I\in\mathcal{G}_s[K]} \left|2^{-sN}\left\langle S^{-1}_{\kappa,\eta} f, h^{n-1}_{I;\kappa}\right\rangle\right|^2 \lesssim 2^{-2rN}2^{2s-w}\|f\|^2_{L^2(\mathbb{R}^{n-1})}.$$

Step 2(b): local average expanded L^4 estimate

We begin by using (9.35) to estimate the $L^4\left(P^K_{s,w}[r]\right)$ norm of $\Lambda^{2s}_{\mathsf{Q}^s_K} f$:

$$\left\|\Lambda^{2s}_{\mathsf{Q}^s_K} f\right\|^4_{L^4\left(P^K_{s,w}[r]\right)} = \int_{P^K_{s,w}[r]} \left|\widehat{(f^s_K)_{\Phi,2s}}(\xi)\right|^4 d\xi = \int_{P^K_{s,w}[r]} \left|\sum_{I\in\mathcal{G}_s[K]} \widehat{(f^s_K)^I_{\Phi,2s}}(\xi)\right|^4 d\xi$$

$$\lesssim 2^{-4(r+s)N} \int_{P^K_{s,w}[r]} \left|\sum_{I\in\mathcal{G}_s[K]} \widehat{\partial^N f^I_{\Phi,2s}}(\xi)\right|^4 d\xi = 2^{-4(r+s)N} \int_{P^K_{s,w}[r]} \left|\sum_{I,J\in\mathcal{G}_s[K]} \widehat{\partial^N f^I_{\Phi,2s}}(\xi)\,\widehat{\partial^N f^J_{\Phi,2s}}(\xi)\right|^2 d\xi$$

$$= 2^{-4(r+s)N} \int_{P^K_{s,w}[r]} \left|\sum_{I,J\in\mathcal{G}_s[K]} \partial^N f^I_{\Phi,2s} \widehat{* \partial^N} f^J_{\Phi,2s}\widehat{\partial}(\xi)\right|^2 d\xi.$$

Then we run the argument in Subsection 5.2, with notation as used there, with the above estimate up until the estimate for $\Omega_t = \Omega_t[K]$, where $2^{-t} \approx \operatorname{dist}(I,J)$ for $I,J \in \mathcal{G}_s[K]$, i.e. $2^{-t} \lesssim \ell(K) = 2^{w-s}$ or $s-w \le t \le s$. Then for $s-w < t < s$ we have

$$\Omega_t[K] \lesssim 2^{-4rN} \sum_{I,J\in\mathcal{G}_s[K]:\ \operatorname{dist}(I,J)\approx 2^{-t}} 2^{-s(n-2)}2^t \left|\left\langle (S_{\kappa,\eta})^{-1} f, h_{I;\kappa}\right\rangle \left\langle (S_{\kappa,\eta})^{-1} f, h_{J;\kappa}\right\rangle\right|^2$$

$$\lesssim 2^{-4rN}2^{-s(n-2)}2^t \sum_{I,J\in\mathcal{G}_s[K]:\ \operatorname{dist}(I,J)\approx 2^{-t}} \left|\left\langle (S_{\kappa,\eta})^{-1} f, h_{I;\kappa}\right\rangle\right|^4$$

$$\lesssim 2^{-4rN}2^{-s(n-2)}2^t 2^{(s-t)(n-1)} \sum_{I\in\mathcal{G}_s[K]} \left|\left\langle (S_{\kappa,\eta})^{-1} f, h_{I;\kappa}\right\rangle\right|^4 \approx 2^{-4rN}2^{-t(n-2)}2^{-s(n-2)} \left\|(\mathsf{Q}^s_K)^{\spadesuit} f\right\|^4_{L^4(S)},$$

which gives

$$\sum_{t=s-w}^{s} \Psi_t[K] \lesssim \sum_{t=s-w}^{s} \Omega_t[K] \lesssim 2^{-4rN} \sum_{t=s-w}^{s} 2^{-t(n-2)}2^{-s(n-2)} \left\|(\mathsf{Q}^s_K)^{\spadesuit} f\right\|^4_{L^4(S)}$$

$$\approx 2^{-4rN}2^{-(s-w)(n-2)}2^{-s(n-2)} \left\|(\mathsf{Q}^s_K)^{\spadesuit} f\right\|^4_{L^4(S)} \lesssim 2^{-4rN}2^{-(2s-w)(n-2)}\|f\|^4_{L^4(S)}.$$

Similarly we obtain

$$\Psi \lesssim 2^{-4rN}2^{-(2s-w)(n-2)}\|f\|^4_{L^4(S)},$$

and adding these results gives,

$$\mathbb{E}^{\mu}_{2^{\mathcal{G}}} \left\|\Lambda^{2s}_{\mathsf{Q}^s_K} f\right\|^4_{L^4(\lambda_n)} \lesssim 2^{-4rN}2^{-(2s-w)(n-2)}\|f\|^4_{L^4(S)}.$$

Step 2(c): local expanded interpolation

Collecting the bounds (9.30) and (9.31) gives,

$$\begin{aligned}\left\|\Lambda_{\mathsf{Q}_K^s}^{2s}f\right\|_{L^2\left(\left|\widehat{\psi}\right|^2\lambda_n\right)}^2 &\lesssim 2^{-2rN}2^{\frac{2s-w}{2}}\left\|f\right\|_{L^2(K)}^2,\\ \mathbb{E}_{2^{\mathcal{G}}}^{\mu}\left\|\Lambda_{\mathsf{Q}_K^s}^{2s}f\right\|_{L^4(\lambda_n)} &\lesssim 2^{-4rN}2^{-\frac{2s-w}{2}\frac{n-2}{2}}\left\|f\right\|_{L^4(S)}^4.\end{aligned}$$

Now we claim that an application of the interpolation Lemma 34 yields,

$$\mathbb{E}_{2^{\mathcal{G}}}^{\mu}\left\|\Lambda_{\mathsf{Q}_K^s}^{2s}f\right\|_{L^p\left(\left|\widehat{\psi}\right|^2\lambda_n\right)}\lesssim 2^{-rN}2^{-(2s-w)\varepsilon_{p,n}'}\left\|f\right\|_{L^p(\mathbb{R}^{n-1})}.$$

Indeed, the calculation at the end of the proof of Lemma 34 shows that if $p>\frac{2n}{n-1}$, then (with notation as in that proof) $\theta=\frac{4}{p}-1$ and so

$$\begin{aligned}&\left[2^{-rN}2^{-\frac{2s-w}{2}\frac{n-2}{2}}\right]^{1-\theta}\left[2^{-rN}2^{\frac{2s-w}{2}}\right]^{\theta}=2^{-rN}2^{-\frac{2s-w}{2}\frac{n-2}{2}}2^{\left(\frac{2s-w}{2}+\frac{2s-w}{2}\frac{n-2}{2}\right)\theta}\\ =\ &2^{-rN}2^{-\frac{2s-w}{2}\frac{n-2}{2}}2^{\left(\frac{2s-w}{2}\frac{n}{2}\right)\theta}=2^{-rN}2^{-(2s-w)\varepsilon_{p,n}'},\end{aligned}$$

where

$$\begin{aligned}\varepsilon_{p,n}' &\equiv \frac{1}{2s-w}\left\{\frac{2s-w}{2}\frac{n-2}{2}-\left(\frac{2s-w}{2}\frac{n}{2}\right)\left(\frac{4}{p}-1\right)\right\}\\ &= \frac{n-2}{4}-\frac{n}{4}\left(\frac{4}{p}-1\right)=\frac{n-1}{2}-\frac{n}{p}=\frac{n-1}{2p}\left(p-\frac{2n}{n-1}\right).\end{aligned}$$

This completes our proof of (9.33) in *Step 2*.

9.4.3. *Step 3: local deterministic argument.* We use Khintchine's inequalities to recast (9.28) as a local square function estimate,

$$\left\|\mathcal{S}_{T,s}^{K,\eta}f\right\|_{L^p\left(T_{s,w}^{K,\natural}\right)}^p\lesssim 2^{-(2s-w)p\varepsilon_{p,n}}\left\|f\right\|_{L^p(\mathbb{R}^{n-1})}^p,\qquad\text{for }p>\frac{2n}{n-1},\tag{9.36}$$

where the local square function $\mathcal{S}_{T,s}^{K,\eta}$ is given by,

$$\mathcal{S}_{T,s}^{K,\eta}f\equiv\mathcal{S}_{T,s}^{\eta}\left(\mathsf{Q}_K^s\right)^{\spadesuit}f=\left(\sum_{I\in\mathcal{G}_s[K]}\left|T\bigtriangleup_{I;\kappa}^{n-1,\eta}f\right|^2\right)^{\frac{1}{2}},\tag{9.37}$$

and where $\mathcal{S}_{T,s}^{\eta}$ is defined in (9.26). Note also that,

$$\left|\mathcal{S}_{T,s}^{\eta}f\right|^2\equiv\sum_{I\in\mathcal{G}_s[U]}\left|T\bigtriangleup_{I;\kappa}^{n-1,\eta}f\right|^2=\sum_{K\in\mathcal{G}_{s-w}[U]}\sum_{I\in\mathcal{G}_s[K]}\left|T\bigtriangleup_{I;\kappa}^{n-1,\eta}f\right|^2=\sum_{K\in\mathcal{G}_{s-w}[U]}\left|\mathcal{S}_{T,s}^{K,\eta}f\right|^2.\tag{9.38}$$

We have $A_+\left(0,2^{2s-w}\right)\subset\bigcup_{K\in\mathcal{G}_{s-w}[U]}T_{s,w}^{K,\natural}\subset A_+^*\left(0,2^{2s-w}\right)$, where the union has bounded overlap C_{lap}, and $A_+^*\left(0,2^{2s-w}\right)$ is a fixed expansion of the quarter annulus $A_+\left(0,2^{2s-w}\right)$. Let $g\in L^{p'}\left(A_+\left(0,2^{2s-w}\right)\right)$

with $\|g\|_{L^{p'}(A_+(0,2^{2s-w}))}=1$. Then we have

$$\sum_{K\in\mathcal{G}_{s-w}[U]}\int_{T^{K,\natural}_{s,w}}\left|\mathcal{S}^{K,\eta}_{T,s}f\right|g \le \sum_{K\in\mathcal{G}_{s-w}[U]}\left\|\mathcal{S}^{K,\eta}_{T,s}f\right\|_{L^p\left(T^{K,\natural}_{s,w}\right)}\|g\|_{L^{p'}\left(T^{K,\natural}_{s,w}\right)}$$
$$\le \left(\sum_{K\in\mathcal{G}_{s-w}[U]}\left\|\mathcal{S}^{K,\eta}_{T,s}f\right\|^p_{L^p\left(T^{K,\natural}_{s,w}\right)}\right)^{\frac{1}{p}}\left(\sum_{K\in\mathcal{G}_{s-w}[U]}\|g\|^{p'}_{L^{p'}\left(T^{K,\natural}_{s,w}\right)}\right)^{\frac{1}{p'}}$$
$$\lesssim \left(\sum_{K\in\mathcal{G}_{s-w}[U]}2^{-(2s-w)p\varepsilon_{p,n}}\left\|(\mathsf{Q}^s_K)^{\spadesuit}f\right\|^p_{L^p(\mathbb{R}^{n-1})}\right)^{\frac{1}{p}}\left(C_{\mathrm{lap}}\|g\|^{p'}_{L^{p'}\left(A^*_+(0,2^{2s-w})\right)}\right)^{\frac{1}{p'}}$$
$$\lesssim C^{\frac{1}{p'}}_{\mathrm{lap}}2^{-(2s-w)\varepsilon_{p,n}}\left\|(\mathsf{Q}^s_U)^{\spadesuit}f\right\|_{L^p(\mathbb{R}^{n-1})}\|g\|_{L^{p'}\left(A^*_+(0,2^{2s-w})\right)}\approx 2^{-(2s-w)\varepsilon_{p,n}}\|f\|_{L^p(\mathbb{R}^{n-1})}\ .$$

9.4.4. *Step 4: expanded deterministic argument.* From (9.33) we have the estimate

$$\mathbb{E}^{\mu}_{2^{\mathcal{G}_s[U]}}\left\|T\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}^s_K\right)^{\spadesuit}f\right\|^p_{L^p\left(P^K_{s,w}[r]\right)}\lesssim 2^{-rp\left(N-\frac{n-1}{p'}\right)}2^{-(2s-w)p\varepsilon_{p,n}}\|f\|^p_{L^p(\mathbb{R}^{n-1})},$$

where $P^{K,\natural}_{s,w}[r]$ is the expanded pipe corresponding to the tube $T^{K,\natural}_{s,w}$. By Khintchine's inequalities this is equivalent to the square function estimate,

$$\left\|\mathcal{S}^{K,\eta}_{T,s}f\right\|^p_{L^p\left(P^K_{s,w}[r]\right)}\lesssim 2^{-rp\left(N-\frac{n-1}{p'}\right)}2^{-(2s-w)p\varepsilon_{p,n}}\|f\|^p_{L^p(\mathbb{R}^{n-1})}\ ,\qquad \text{for } p>\frac{2n}{n-1}, \tag{9.39}$$

where $\mathcal{S}^{K,\eta}_{T,s}f$ is the local square function defined in (9.37). The only difference between the left hand sides of the inequalities (9.36) and (9.39), is that the second inequality is integrated over the expanded pipe $P^K_{s,w}[r]$ instead of the tube $T^{K,\natural}_{s,w}$. As a consequence, we obtain by following the previous argument with $\|g\|_{L^{p'}(A_+(0,2^{2s-w}))}=1$ that,

$$\sum_{K\in\mathcal{G}_{s-w}[U]}\int_{P^K_{s,w}[r]}\left|\mathcal{S}^{K,\eta}_{T,s}f\right|g \le \sum_{K\in\mathcal{G}_{s-w}[U]}\left\|\mathcal{S}^{K,\eta}_{T,s}f\right\|_{L^p\left(P^K_{s,w}[r]\right)}\|g\|_{L^{p'}\left(P^K_{s,w}[r]\right)}$$
$$\le \left(\sum_{K\in\mathcal{G}_{s-w}[U]}\left\|\mathcal{S}^{K,\eta}_{T,s}f\right\|^p_{L^p\left(P^K_{s,w}[r]\right)}\right)^{\frac{1}{p}}\left(\sum_{K\in\mathcal{G}_{s-w}[U]}\|g\|^{p'}_{L^{p'}\left(P^K_{s,w}[r]\right)}\right)^{\frac{1}{p'}}$$
$$\lesssim \left(\sum_{K\in\mathcal{G}_{s-w}[U]}2^{-rp\left(N-\frac{n-1}{p'}\right)}2^{-(2s-w)p\varepsilon_{p,n}}\left\|(\mathsf{Q}^s_K)^{\spadesuit}f\right\|^p_{L^p(\mathbb{R}^{n-1})}\right)^{\frac{1}{p}}\left(C_{\mathrm{lap}}2^{r(n-1)}\|g\|^{p'}_{L^{p'}\left(A^*_+(0,2^{2s-w})\right)}\right)^{\frac{1}{p'}}$$
$$\lesssim C^{\frac{1}{p'}}_{\mathrm{lap}}2^{-r\left(N-2\frac{n-1}{p'}\right)}2^{-(2s-w)\varepsilon_{p,n}}\left\|(\mathsf{Q}^s_U)^{\spadesuit}f\right\|_{L^p(\mathbb{R}^{n-1})}\|g\|_{L^{p'}\left(A^*_+(0,2^{2s-w})\right)}\lesssim 2^{-r\delta}2^{-(2s-w)\varepsilon_{p,n}}\|f\|_{L^p(\mathbb{R}^{n-1})}\ ,$$

since $\|g\|_{L^{p'}\left(A^*_+(0,2^{2s-w})\right)}=1$, where $\delta=N-2\frac{n-1}{p'}>0$ for N chosen sufficiently large.

Now sum in r to obtain

$$\int_{A_+(0,2^{2s-w})}\left(\sum_{K\in\mathcal{G}_{s-w}[U]}\left|\mathcal{S}^{K,\eta}_{T,s}f\right|\right)g\lesssim \sum_{K\in\mathcal{G}_{s-w}[U]}\sum_{r=0}^{s-w}\int_{P^K_{s,w}[r]}\left|\mathcal{S}^{K,\eta}_{T,s}f\right|g=\sum_{r=0}^{s-w}\sum_{K\in\mathcal{G}_{s-w}[U]}\int_{P^K_{s,w}[r]}\left|\mathcal{S}^{K,\eta}_{T,s}f\right|g$$
$$\lesssim \sum_{r=0}^{s-w}2^{-r\delta}2^{-(2s-w)\varepsilon_{p,n}}\left\|(\mathsf{Q}^s_U)^{S_{\kappa,\eta}}f\right\|_{L^p(\mathbb{R}^{n-1})}\lesssim 2^{-(2s-w)\varepsilon_{p,n}}\|f\|_{L^p(\mathbb{R}^{n-1})}\ .$$

If we take the supremum over g as above, i.e. $g\in L^{p'}\left(A_+\left(0,2^{2s-w}\right)\right)$ with $\|g\|_{L^{\left(\frac{p}{2}\right)'}(A_+(0,2^{2s-w}))}=1$, we obtain

$$\left\|\sum_{K\in\mathcal{G}_{s-w}[U]}\left|\mathcal{S}^{K,\eta}_{T,s}f\right|\right\|_{L^p(A_+(0,2^{2s-w}))}\lesssim 2^{-(2s-w)\varepsilon_{p,n}}\|f\|_{L^p(\mathbb{R}^{n-1})}\ .$$

Then we have from (9.38) that

$$\begin{aligned}\left\|\mathcal{S}_{T,s}^{\eta}f\right\|_{L^{p}(A_{+}(0,2^{2s-w}))} &= \left\|\left(\sum_{K\in\mathcal{G}_{s-w}[U]}\left|\mathcal{S}_{T,s}^{K,\eta}f\right|^{2}\right)^{\frac{1}{2}}\right\|_{L^{p}(A_{+}(0,2^{2s-w}))}\\ \leq \left\|\sum_{K\in\mathcal{G}_{s-w}[U]}\left|\mathcal{S}_{T,s}^{K,\eta}f\right|\right\|_{L^{p}(A_{+}(0,2^{2s-w}))} &\lesssim 2^{-(2s-w)\varepsilon_{p,n}}\left\|f\right\|_{L^{p}(\mathbb{R}^{n-1})}\ ,\end{aligned}$$

which gives (9.27) since $2^{-(2s-w)\varepsilon_{p,n}} \leq 2^{-s\varepsilon_{p,n}}$ for $0 \leq w \leq s$.

This completes the proof of (9.24).

9.5. **Control of the** lower distal **form.** The pairs (I,J) arising in the lower distal form,

$$\mathsf{B}_{\text{distal}}^{\text{lower}}(f,g) = \sum_{d<0}\sum_{k\in\mathbb{Z}}\mathsf{B}_{\text{distal}}^{k,d}(f,g),$$

where

$$\begin{aligned}\mathsf{B}_{\text{distal}}^{k,d}(f,g) &\equiv \sum_{(I,J)\in\mathcal{X}^{k,d}}\left\langle T\,\triangle_{I;\kappa}^{n-1,\eta}f,\triangle_{J;\kappa}^{n,\eta}g\right\rangle,\\ \text{where } \mathcal{X}^{k,d} &\equiv \left\{(I,J)\in\mathcal{X} : \ell(J)=2^{k}, \text{ and } 2^{d}\leq\ell(I)^{2}\operatorname{dist}(0,J)\leq 2^{d+1}\right\},\\ \text{and } \mathcal{X} &\equiv \left\{(I,J)\in\mathcal{G}[U]\times\mathcal{D} : 2^{m+1}I\subset S \text{ and } \pi_{\tan}(J)\cap\Phi(2U)=\emptyset\right\},\end{aligned}$$

were already included in the arguments given to control the lower disjoint form in the preceding three subsections, such as those used to prove (9.33).

9.6. **Wrapup.** Combining all of the estimates in this section, and taking into account the change of parameters from m,d to w,r for the estimates with $d<0$, we see that we have obtained the desired bound for the disjoint form $\mathsf{B}_{\text{disjoint}}(f,g) = \mathsf{B}_{\text{disjoint}}^{\text{upper}}(f,g) + \mathsf{B}_{\text{disjoint}}^{\text{lower}}(f,g)$,

$$\begin{aligned}\left|\mathsf{B}_{\text{disjoint}}^{\text{upper}}(f,g)\right| &= \left|\sum_{d\geq 0}\sum_{m=1}^{\infty}\sum_{k\in\mathbb{Z}}\mathsf{B}_{\text{disjoint}}^{k,d,m}(f,g)\right| \lesssim \|f\|_{L^{p}}\|g\|_{L^{p'}}\ ,\\ \mathbb{E}_{2^{\mathcal{G}}}^{\mu}\left|\mathsf{B}_{\text{disjoint}}^{\text{lower}}(f,g)\right| &= \mathbb{E}_{2^{\mathcal{G}}}^{\mu}\left|\sum_{d<0}\sum_{m=1}^{\infty}\sum_{k\in\mathbb{Z}}\mathsf{B}_{\text{disjoint}}^{k,d,m}(f,g)\right| \lesssim \|f\|_{L^{p}}\|g\|_{L^{p'}}\ ,\end{aligned}$$

where if $(I,J)\in\mathcal{P}_{m}^{k,d}$ and $\ell(I)=2^{-s}$ in the above sum, it is understood that $-s\leq d<\infty$.

We see also that we have obtained the desired bound for the distal form $\mathsf{B}_{\text{distal}}(f,g) = \mathsf{B}_{\text{distal}}^{\text{upper}}(f,g) + \mathsf{B}_{\text{distal}}^{\text{lower}}(f,g)$,

$$\begin{aligned}\left|\mathsf{B}_{\text{distal}}^{\text{upper}}(f,g)\right| &\lesssim \|f\|_{L^{p}}\|g\|_{L^{p'}}\ ,\\ \mathbb{E}_{2^{\mathcal{G}}}^{\mu}\left|\mathsf{B}_{\text{distal}}^{\text{lower}}(f,g)\right| &\lesssim \|f\|_{L^{p}}\|g\|_{L^{p'}}\ .\end{aligned}$$

Finally, we also have the norm expectation (9.24),

$$\mathbb{E}_{2^{\mathcal{G}[U]}}^{\mu}\left\|T\left(\mathcal{A}_{\mathbf{a}}\mathsf{Q}_{U}^{s}\right)^{\spadesuit}f\right\|_{L^{p}(B(0,2^{2s}))} \lesssim 2^{-\varepsilon_{n,p}s}\|f\|_{L^{p}}^{p}\ ,$$

for $p>\frac{2n}{n-1}$, which will play a critical role in completing the proof of our main theorem in the next section.

10. Completion of the proof of the probabilisitic extension Theorem 4

Consider the norm $\left\| \left(\widehat{(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f} \right)_{\Phi,2s} \right\|_{L^p\left(\mathbf{1}_{\mathbb{R}^n \setminus B(0,2^{2s})}\lambda_n\right)}$ for each fixed $f \in L^p$, $s \in \mathbb{N}$ and $a \in \mathbf{a}$, and choose $g_{f,s,a} \in L^{p'}(\lambda_n)$ such that

$$\triangle_{J;\kappa} g_{f,s,a} = 0 \text{ for } J \in \mathcal{D}\left[B\left(0,2^{2s}\right)\right], \tag{10.1}$$

$$\left\| \left(\widehat{(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f} \right)_{\Phi,2s} \right\|_{L^p\left(\mathbf{1}_{\mathbb{R}^n \setminus B(0,2^{2s})}\lambda_n\right)} = \left| \left\langle T\left((\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f \right)_{2s}, g_{f,s,a} \right\rangle \right| \text{ and } \|g_{f,s,a}\|_{L^{p'}(\lambda_n)} = 1.$$

Since $\mathsf{B}_{\text{disjoint}}^{\text{lower}}\left((\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f, g_{f,s,\mathbf{a}} \right)$ and $\mathsf{B}_{\text{distal}}^{\text{lower}}\left((\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f, g_{f,s,\mathbf{a}} \right)$ each vanish by the assumption on the Alpert support of $g_{f,s,a}$ in (10.1), and the definitions of the lower disjoint and distal forms, we have

$$\begin{aligned}
&\mathbb{E}^{\mu}_{2^{\mathcal{G}[S]}} \left| \left\langle T\left((\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f \right)_{2s}, g_{f,s,\mathbf{a}} \right\rangle \right| = \mathbb{E}^{\mu}_{2^{\mathcal{G}_s[S]}} \left| \left\langle T\mathcal{A}\left((\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f \right)_{2s}, g_{f,s,\mathbf{a}} \right\rangle \right| \\
= \; & \mathbb{E}^{\mu}_{2^{\mathcal{G}_s[S]}} \Big| \mathsf{B}_{\text{below}}\left(T\left((\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f \right)_{2s}, g_{f,s,a} \right) + \mathsf{B}_{\text{above}}\left(T\left((\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f \right)_{2s}, g_{f,s,\mathbf{a}} \right) \\
& \qquad + \mathsf{B}_{\text{disjoint}}^{\text{upper}}\left(T\left((\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f \right)_{2s}, g_{f,s,\mathbf{a}} \right) + \mathsf{B}_{\text{distal}}^{\text{upper}}\left(T\left((\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f \right)_{2s}, g_{f,s,\mathbf{a}} \right) \Big\} \\
\lesssim \; & \sup_{\mathbf{a}} 2^{-\varepsilon_{n,p} s} \left\| (\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f \right\|_{L^p} \|g_{f,s,\mathbf{a}}\|_{L^{p'}} ,
\end{aligned}$$

from estimates proved in previous sections. Thus we conclude from this and (9.24) that

$$\begin{aligned}
&\mathbb{E}^{\mu}_{2^{\mathcal{G}[S]}} \left\| T\left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s \right)^{\spadesuit} f \right\|_{L^p} \\
\lesssim \; & \mathbb{E}^{\mu}_{2^{\mathcal{G}_s[S]}} \left\| \left(\widehat{(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f} \right)_{\Phi,2s} \right\|_{L^p\left(\mathbf{1}_{\mathbb{R}^n \setminus B(0,2^{2s})}\lambda_n\right)} + \mathbb{E}^{\mu}_{2^{\mathcal{G}_s[S]}} \left\| \left(\widehat{(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f} \right)_{\Phi,2s} \right\|_{L^p(B(0,2^{2s}))} \\
= \; & \mathbb{E}^{\mu}_{2^{\mathcal{G}_s[S]}} \left| \left\langle T\left((\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f \right)_{2s}, g_{f,s,\mathbf{a}} \right\rangle \right| + \mathbb{E}^{\mu}_{2^{\mathcal{G}_s[S]}} \left\| \left(\widehat{(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f} \right)_{\Phi,2s} \right\|_{L^p(B(0,2^{2s}))} \\
\lesssim \; & \sup_{\mathbf{a}} 2^{-\varepsilon_{n,p} s} \left\| (\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit} f \right\|_{L^p} \|g_{f,s,\mathbf{a}}\|_{L^{p'}} + 2^{-\varepsilon_{n,p} s} \left\| (\mathsf{Q}_U^s)^{\spadesuit} f \right\|_{L^p} \lesssim 2^{-\varepsilon_{n,p} s} \|f\|_{L^p} ,
\end{aligned}$$

since the multipliers $(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s)^{\spadesuit}$ and the conjugated projection $(\mathsf{Q}_U^s)^{\spadesuit}$ are both bounded on L^p by the square function estimates (2.1). Finally we have

$$\begin{aligned}
&\mathbb{E}^{\mu}_{2^{\mathcal{G}[S]}} \left\| T\left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s \right)^{\spadesuit} f \right\|_{L^p} = \mathbb{E}^{\mu}_{2^{\mathcal{G}[S]}} \left\| \sum_{s=1}^{\infty} T\left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s \right)^{\spadesuit} f \right\|_{L^p} \\
\leq \; & \sum_{s=1}^{\infty} \mathbb{E}^{\mu}_{2^{\mathcal{G}[S]}} \left\| T_S\left(\mathcal{A}_{\mathbf{a}} \mathsf{Q}_U^s \right)^{\spadesuit} f \right\|_{L^p} \leq \sum_{s=1}^{\infty} 2^{-\varepsilon_{n,p} s} \|f\|_{L^p} \lesssim \|f\|_{L^p} .
\end{aligned}$$

This completes the proof of (1.9), and hence that of Theorem 4.

11. Concluding remarks

The two weight testing methods used in this paper might also be applicable to the following open *probabilistic* problems:

(1) proving a probabilistic analogue of the Bochner-Riesz conjecture or even the stronger local smoothing conjecture. In the context of the (nonprobabilistic) extension conjecture, see Sogge [Sog] for a proof that local smoothing implies Bochner-Riesz, and Tao [Tao1] for a proof that Bochner-Riesz implies Fourier restriction,
(2) replacing the sphere in Theorem 4 with any smooth surface of nonvanishing Gaussian curvature, and possibly with appropriate smooth surfaces of finite type (and with altered indices p),
(3) replacing the Fourier kernel $e^{-ix\cdot\xi}$ in Theorem 4 with a more general kernel $\Omega(x,\xi)$,
(4) to multilinear probabilistic variants of the extension conjecture,
(5) deciding the endpoint case $q = p' \frac{n+1}{n-1}$ when $2 < p < \frac{2n}{n-1}$ in (1.4),

(6) and finally to the much more challenging problem of boundedness of the maximal spherical partial sum operator in a probabilistic sense.

The main open problem is of course the full *deterministic* Fourier extension conjecture (1.1).

References

[AlSaUr] M. Alexis, E. Sawyer and I. Uriarte-Tuero, *A $T1$ theorem for general Calderón-Zygmund operators with doubling weights, and optimal cancellation conditions, II*, Journal of Functional Analysis **285**, issue 11, (2023), arXiv:2111.06277.

[AlLuSa] M. Alexis, J.-L. Luna-Garcia, and E. Sawyer, *Haar basis and frame testing*, arXiv:2309.03743.

[Alp] Bradley K. Alpert, *A class of bases in L^2 for the sparse representation of integral operators*, SIAM J. Math. Anal 1 (1993), p. 246-262.

[Bou1] J. Bourgain, *Besicovitch type maximal operators and applications to Fourier analysis*, Geom. Funct. Anal. **1** (1990), 147-187.

[Bou] J. Bourgain, *Some new estimates for oscillatory integrals*, in "Essays on Fourier analysis in honor of Elias M. Stein", ed. C. Fefferman, R. Fefferman and S. Wainger, Princeton University Press, 1994.

[BoGu] J. Bourgain and L. Guth, *Bounds on oscillatory integral operators based on multilinear estimates* (2018), arXiv:1012.3760.

[CaSj] L. Carleson and P. Sjölin, *Oscillatory integrals and a multiplier problem for the disc*, Studia Math. **44** (1972), 287–299.

[CaHaLa] Peter G. Cassaza, Deguang Han and David R. Larsen, *Frames for Banach spaces,* Contemporary Mathematics **247** (1999), 149-182.

[DaJo] David, Guy, Journé, Jean-Lin, *A boundedness criterion for generalized Calderón-Zygmund operators,* Ann. of Math. (2) **120** (1984), 371–397, MR763911 (85k:42041).

[Dem] Ciprian Demeter, *Decouplings and applications*, Proc. Int. Cong. of Math. - 2018 Rio de Janeiro, Vol **2** (1535-1556).

[Fef] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Mathematica **124** (1970), no. 1, 9-36.

[Fef2] C. Fefferman, *The multiplier problem for the ball*, Annals of Math. **94** (1971), 330-336.

[FeSt] C. Fefferman and E.M. Stein, *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107-115.

[Gut] Larry Guth, *A restriction estimate using polynomial partitioning*, J. Amer. Math. Soc., **29** (2016), 371–413.

[HiRo] J. Hickman and K. M. Rogers, Improved Fourier restriction estimates in higher dimensions, arXiv:1807.10940v3.

[Hyt] Tuomas Hytönen, *The two weight inequality for the Hilbert transform with general measures,* Proc. London Math. Soc., Vol. **117**, issue 3 (2018), 483-526.

[Hyt1] Tuomas Hytönen, *Martingales and harmonic analysis*, Lecture notes of a course at the University of Helsinki, Autumn 2012.

[HyVu] Tuomas Hytönen and Emil Vuorinen, *A two weight inequality between $L^p(\ell^2)$ and L^p*, Mathematika **64**, issue 1, (2018), 284-302.

[IoZh] Alex Iosevich and Ruixiang Zhang, *A distinction between the paraboloid and the sphere in weighted restriction, arXiv:2312.12779*.

[KaLaTa] Nets Katz, Izabella Laba and Terence Tao, *An improved bound on the Minkowski dimension of Besicovitch sets in $\mathbb{R}^3$*, Ann. of Math. (2) **152** (2000), 383-446.

[LaTa] Izabella Laba and Terence Tao, *An improved bound for the Minkowski dimension of Besicovitch sets in medium dimension*, https://arxiv.org/abs/math/0004015v1.

[LaWa] Izabella Laba and Hong Wang, *Decoupling and near-optimal restriction estimates for Cantor sets*, Int. Math. Res. Not. (2018), 2944-2966.

[Lac] Michael T. Lacey, *Two weight inequality for the Hilbert transform: A real variable characterization, II*, Duke Math. J. Volume **163**, Number 15 (2014), 2821-2840.

[LaSaShUr3] Michael T. Lacey, Eric T. Sawyer, Chun-Yen Shen, and Ignacio Uriarte-Tuero, *Two weight inequality for the Hilbert transform: A real variable characterization I*, Duke Math. J, Volume **163**, Number 15 (2014), 2795-2820.

[LaSaUr1] Lacey, Michael T., Sawyer, Eric T., Uriarte-Tuero, Ignacio, *A characterization of two weight norm inequalities for maximal singular integrals with one doubling measure,* Analysis & PDE, Vol. **5** (2012), No. 1, 1-60.

[LaWi] Lacey, Michael T., Wick, Brett D., *Two weight inequalities for Riesz transforms: uniformly full dimension weights*, arXiv:1312.6163v3.

[Mat] Pertti Mattila, *Fourier analysis and Hausdorff dimension*, Cambridge University Press.

[MoVaVe] A. Moyua, A. Vargas, and L. Vega, *Schrödinger maximal function and restriction properties of the Fourier transform*, Internat. Math. Res. Notices (1996), no. **16**, 793–815.

[MuOl] C. Muscalu and I. Oliveira, *A new approach to the Fourier extension problem for the paraboloid*, arXiv:2110.01482v4.

[NTV4] F. Nazarov, S. Treil and A. Volberg, *Two weight estimate for the Hilbert transform and corona decomposition for non-doubling measures*, preprint (2004) arXiv:1003.1596.

[Nic] Fabio Nicola, *Slicing surfaces and the Fourier restriction conjecture*, Proceedings of the Edinburgh Mathematical Society, **52** (2009), pp 515-527, doi:10.1017/S0013091507000995.

[RaSaWi] Robert Rahm, Eric T. Sawyer and Brett D. Wick, *Weighted Alpert wavelets*, Journal of Fourier Analysis and Applications (IF1.**273**), Pub Date : 2020-11-23, DOI: 10.1007/s00041-020-09784-0, arXiv:1808.01223v2.

[Saw] Eric T. Sawyer, *A characterization of a two weight norm inequality for maximal operators*, Studia Math., 75 (1982), 1–11.

[Saw3] E. Sawyer, *A characterization of two weight norm inequalities for fractional and Poisson integrals*, Trans. A.M.S. **308** (1988), 533-545, MR{930072 (89d:26009)}.

[Saw6] E. Sawyer, *A T1 theorem for general Calderón-Zygmund operators with comparable doubling weights and optimal cancellation conditions*, Journal d'Analyse Mathématique **146** no. 1 (2022), 205-297.

[SaShUr7] Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A *two weight theorem for α-fractional singular integrals with an energy side condition*, Revista Mat. Iberoam. **32** (2016), no. 1, 79-174.

[SaShUr12] Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, *A two weight local Tb theorem for the Hilbert transform*, Revista Mat. Iberoam **37** (2021), No. 2, 415-641.

[SaWi] Eric T. Sawyer and Brett D. Wick, *Two weight L^p inequalities for smooth Calderón-Zygmund operators and doubling measures*, arXiv:2211.01920v4.

[Sog] C. Sogge, *Propagation of singularities and maximal functions in the plane*, Inv. Math. **104** (1991), 349-376.

[Sted] Richard James Stedman, *The Restriction and Kakeya Conjectures*, etheses.bham.ac.uk/id/eprint/5466/1/ Stedman14MPhil.pdf.

[Ste] E. M. Stein, *Some problems in harmonic analysis*, Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1, pp. 3-20, Proc. Sympos. Pure Math., **XXXV**, Part, Amer. Math. Soc., Providence, R.I., 1979.

[Ste2] E. M. Stein, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, N. J., 1993.

[Sto] Betsy Stovall, *Waves, Spheres, and Tubes: A Selection of Fourier Restriction Problems, Methods, and Applications*, Notices of the A.M.S. **66**, No. 7 (2019), 1013-1022.

[Tao] Terence Tao, S*ome recent progress on the restriction conjecture,* arXiv:math/0303136, In: Brandolini, L., Colzani, L., Travaglini, G., Iosevich, A. (eds) Fourier Analysis and Convexity. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston, MA. https://doi.org/10.1007/978-0-8176-8172-2_10.

[Tao1] Terence Tao, *The Bochner-Riesz conjecture implies the Restriction conjecture,* Duke Math. J. **96** (1999), 363-376.

[Tao2] Terence Tao, 245C, Notes 1: Interpolation of L^p spaces | What's new.

[Tao3] Terence Tao, 247B, Notes 1: Restriction theory.

[Tao4] Terence Tao, *A sharp bilinear restrictions estimate for paraboloids*, Geom. Funct. Anal., **13** (6), (2003):1359–1384.

[Tao5] Terence Tao, *Recent developments on the Kakeya and restriction problems,* IPAM tutorial, oide **2001**_4282 Tao Kakeya and restriction.pdf.

[Tao6] Terence Tao, *Restriction theorems, Besicovitch sets, and applications to PDE,* Math 254B - Topics in real analysis, UCLA.

[Tom] P.A. Tomas, *A restriction theorem for the Fourier transform*, Bull. Amer. Math. Soc. **81** (1975), 477–478.

[Vol] A. Volberg, *Calderón-Zygmund capacities and operators on nonhomogeneous spaces,* CBMS, Reg. Conf. Ser. in Math., Number **100** (2003), 167 pages .

[Wan] Hong Wang, *An improved restriction estimate in* $\mathbb{R}^3$, arXiv:2210.03878v2.

[WaZa] Hong Wang and Joshua Zahl, *Volume estimates for unions of convex sets, and the Kakeya set conjecture in three dimensions*, arXiv:2502.17655v1.

[Wol2] T. Wolff, *An improved bound for Kakeya type maximal functions*, Rev. Mat. Iberoamericana **11** (1995), no. 3, 651–674.

[Wol] Thomas Wolff, *Recent work connected with the Kakeya problem*, DOI:10.1090/ulect/029/11Corpus ID: 14402276, (2007).

[Wol3] Thomas Wolff, *Lectures on harmonic analysis*, AMS University lecture series volume **29** (2003).

[Zyg] A. Zygmund, *On Fourier coefficients and transforms of functions of two variables*, Studia Mathematica **50** (1974), no. 2, 189-201.

[blogs.rice] https://bpb-us-e1.wpmucdn.com/blogs.rice.edu/dist/8/4754/files/2016/09/dHHU_Chap2-1mylqq6.pdf

Eric T. Sawyer, Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S 4K1 Canada

E-mail address: sawyer@mcmaster.ca